FIFTY-SEVENTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Precalculus 2013

Prepared by:

Department of Mathematics
The University of Tennessee at Chattanooga
Chattanooga, Tennessee

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, Tennessee

Coordinated by: Andrew Ledoan

Scoring formula: $4R - W + 40$

DIRECTIONS:

I. Do not open this booklet until you are told to do so.

II. This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

III. This test has been constructed so that most of you are not expected to answer all the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

IV. If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

V. When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State Community College, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
TRW Commercial Steering Division, Lebanon, Tennessee
Wright Industries, Inc., Nashville, Tennessee

1. If $f(x) = \sqrt{3 - e^{2x}}$, find the domain of f^{-1}.

   A. $(-\sqrt{3}, \sqrt{3})$ B. $\left(-\infty, \frac{\ln 3}{2}\right]$ C. $[-\sqrt{3}, \sqrt{3}]$ D. $(0, \sqrt{3})$ E. $[0, \sqrt{3})$

2. Solve the exponential equation $x^5e^x + 3x^4e^x = 4x^3e^x$ for x.

   A. $(-\infty, \infty)$ B. $\{-1, 0, 4\}$ C. $\{0\}$ D. $\{-4, 0, 1\}$ E. No solution

3. If $f(x) = 2x^3 + Cx^2 + Dx - 5$, and if $f(2) = 3$ and $f(-2) = -37$, find the value of $C + D$.

   A. -18 B. 0 C. -1 D. -2 E. -34

4. Solve the system of equations $3x - 7y + 11z = 4$, $x - 2y + 3z = 1$ and $-2x + 8y - 16z = 4$ for (x, y, z), using any method.

   A. $(6, 2, 0)$ B. $(1, 3, 2)$ C. $(-2, 2, 1)$ D. $(-6, -11, -5)$ E. $(-2, 4, 2)$

5. A bacteria culture grows exponentially, so that the initial number has doubled in 4 hours. How many times the initial number will be present after 20 hours?

   A. 16 B. 32 C. 64 D. 5 E. 10

6. Simplify $\dfrac{(2x^3y)^{-1}(8x^{-6}y^3)^{1/3}(x^{7/6}y^{-1/3})^6}{(2x^{-1}y^4)^2(49x^{-2}y^4)^{-1/2}(xy^{1/2})^0}$.

   A. $\dfrac{7x^3}{4y^6}$ B. $\dfrac{49x}{4y^4}$ C. $\dfrac{7x^3}{4y^8}$ D. $\dfrac{7x^2}{4y^{17/2}}$ E. $\dfrac{49}{4y^{9/2}}$

7. After 3 days, a sample of Radon-222 has decayed to 58% of its original amount. What is the half-life of Radon-222?

   A. About 3.82 days B. About 1.26 days C. About 4.03 days

   D. About 2.08 days E. About 1.74 days

8. If $\log_a x = 8$, $\log_a y = 6$ and $\log_a z = 4$, find $\log_a \dfrac{\sqrt[4]{y^2z^5}}{\sqrt[4]{x^3z^{-2}}}$.

   A. 8 B. 6 C. 4 D. 7 E. 16

9. Find an equation of the line that is the perpendicular bisector of the line segment that has endpoints $(-1, 2)$ and $(3, 10)$.

   A. $x - 2y = -13$ B. $x + 2y = 13$ C. $x - 2y = -13$

   D. $x - 2y = -11$ E. $x - 2y = 11$

10. If $f(x) = 5 + (3x + 2)e^{2x}$, find $f^{-1}(7)$.

    A. 0 B. 2 C. 5 D. 7 E. 10

11. Assume $\pi < \theta < \dfrac{3\pi}{2}$ and $\cos\theta = -\dfrac{4}{5}$. Find $\sin\theta$.

    A. $-\dfrac{3}{4}$ B. $\dfrac{3}{4}$ C. $-\dfrac{3}{5}$ D. $\dfrac{2}{5}$ E. $\dfrac{4}{5}$

12. Rewrite the expression $\sin(\tan^{-1} x - \sin^{-1} x)$ as an algebraic expression of x that does not involve trigonometric functions.

    A. $\dfrac{x\sqrt{1+x^2} - 1}{\sqrt{1-x^2}}$ B. $\dfrac{x(\sqrt{1+x^2} - 1)}{\sqrt{1-x^2}}$ C. $\dfrac{x^2\sqrt{1-x^2}}{1+x^2}$

    D. $\dfrac{x\sqrt{1-x^2} - 1}{\sqrt{1+x^2}}$ E. $\dfrac{x(\sqrt{1-x^2} - 1)}{\sqrt{1+x^2}}$

13. What is the coefficient of the term containing x^3y^5 in the expansion of the binomial $(2x - 3y)^8$?

    A. 1679616 B. -1944 C. -108864 D. 48384 E. 264

14. Find the last digit in the number 3^{459}.

    A. 9 B. 7 C. 3 D. 1 E. 0

15. Convert the polar equation $r = 2\sin\theta - \cos\theta$ to rectangular form.

    A. $\left(x - \dfrac{1}{2}\right)^2 + (y+1)^2 = \dfrac{5}{4}$ B. $(x-1)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{5}{4}$

    C. $(x-1)^2 + (y-2)^2 = 4$ D. $\left(x + \dfrac{1}{2}\right)^2 + (y-1)^2 = \dfrac{5}{4}$

    E. $(x-1)^2 + \left(y - \dfrac{1}{2}\right)^2 = 2$

16. Find a fourth-degree polynomial function $f(x)$ with real coefficients that has i and $3i$ as zeros and such that $f(-1) = 40$.

A. $f(x) = x^4 - 10x^2 - 9$ B. $f(x) = x^4 + 10x^2 + 9$ C. $f(x) = 2x^4 + 20x^2 + 18$

D. $f(x) = 2x^4 - 20x^2 - 18$ E. $f(x) = 3x^4 + 30x^2 + 27$

17. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{63} 64)$.

A. 16 B. 28 C. 32 D. 8 E. 6

18. Find the solution set of the inequality $\dfrac{2}{3} \le \dfrac{1}{x-2} < 1$.

A. $\left(-\infty, \dfrac{7}{2}\right]$ B. $\left(3, \dfrac{7}{2}\right]$ C. $\left[\dfrac{7}{2}, \infty\right)$ D. $\left[3, \dfrac{7}{2}\right]$ E. $(-\infty, 2)$

19. Find the number of solutions of the trigonometric equation $\sec x \tan x - 4 \sin x = 0$ in the interval $[0, 2\pi)$.

A. 2 B. 3 C. 4 D. 5 E. 6

20. Find the absolute value of $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} + i^{15}$.

A. $\dfrac{2}{5}$ B. $\dfrac{\sqrt{29}}{5}$ C. $\sqrt{2}$ D. $\dfrac{\sqrt{26}}{5}$ E. $\dfrac{3}{5}$

21. Amy has 75 coins. Some of the coins are quarters and the rest are dimes. If the total amount is \$12.90, how many coins of each type does she have?

A. 36 quarters, 39 dimes B. 40 quarters, 29 dimes C 32 quarters, 49 dimes

D. 34 quarters, 44 dimes E. 46 quarters, 14 dimes

22. Given that $a = (\log_{125} 5)^{\log_5 125}$, find the value of $\log_3 a$.

A. $\dfrac{1}{3}$ B. 3 C. -3 D. 5 E. -5

23. Let $f(x) = \sqrt{x^2 + x + 1}$ and $g(x) = x^2 - 1$. Find the range of $f \circ g$.

A. $\left[\dfrac{1}{2}, \infty\right)$ B. $\left[\dfrac{3}{4}, \infty\right)$ C. $[1, \infty)$ D. $\left[\dfrac{\sqrt{3}}{2}, \infty\right)$ E. $(-\infty, \infty)$

24. A tugboat tows a barge 48 miles up the Tennessee River at 10 miles per hour and returns down the river at 12 miles per hour. If the entire trip took 11 hours, what is the rate of the river's current?

A. 12 miles per hour B. 8 miles per hour C. 6 miles per hour

D. 5 miles per hour E. 4 miles per hour

25. Find the sum of the infinite geometric series $\frac{3}{\sqrt{5}} - \frac{6}{5} + \frac{12}{5\sqrt{5}} - \frac{24}{25} + \dots$.

A. $3\sqrt{5} - 2$ B. $3\sqrt{5} - 6$ C. $\frac{48}{25\sqrt{5}}$ D. $\frac{3}{\sqrt{5}}$ E. $2\sqrt{5}$

26. An airplane flies in a straight path for 1 hour and 30 minutes. It then makes a course correction, heading 10° to the right of its original course, and flies 2 hours in the new direction. If the airplane maintains a constant speed of 625 miles per hour, how far is it from its starting point?

A. About 471 miles B. About 2055 miles C. About 2179 miles

D. About 563 miles E. About 664 miles

27. Find all values of x in the interval $[0, 2\pi]$ such that $2\cos x + 1 > 0$.

A. $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ B. $\left[0, \frac{5\pi}{6}\right] \cup \left[\frac{7\pi}{6}, 2\pi\right]$ C. $\left[0, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right]$

D. $\left[0, \frac{5\pi}{6}\right) \cup \left(\frac{7\pi}{6}, 2\pi\right]$ E. $\left[0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right]$

28. If $f(x) = \ln(x + \sqrt{x^2 + 1})$, find $f^{-1}(x)$.

A. $f^{-1}(x) = \frac{e^{2x} - 1}{2e^x}$ B. $f^{-1}(x) = \frac{1 - e^x}{2e^{2x}}$ C. $f^{-1}(x) = \frac{e^x - 1}{e^x}$

D. $f^{-1}(x) = \frac{1}{x + \sqrt{x^2 + 1}}$ E. $f(x)$ does not have an inverse.

29. How many 5-letter words (meaningful or not) contain 3 different consonants and 2 different vowels?

A. 65780 B. 133000 C. 160800 D. 1596000 E. 1659000

30. Find the number $(1-\sqrt{3}i)^5$.

A. $16+32\sqrt{3}i$ B. $16\sqrt{3}+16i$ C. $16+16\sqrt{3}i$ D. $16-32\sqrt{3}i$ E. $16-16\sqrt{3}i$

31. Find the solution set of the inequality $\ln(x^2-2x-2)\leq 0$.

A. $[-1,3]$ B. $(1-\sqrt{3},1+\sqrt{3})$ C. $(-\infty,1-\sqrt{3})\cup(1+\sqrt{3},\infty)$

D. $[-1,1-\sqrt{3})\cup(1+\sqrt{3},3]$ E. No solution

32. Find the term of $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^{12}$ that does not contain x.

A. $\dfrac{231}{16}$ B. $\dfrac{11}{4379}$ C. $\dfrac{40095}{256}$ D. $-\dfrac{891}{16}$ E. $\dfrac{55}{144}$

33. Find the eccentricity of the ellipse $4x^2+25y^2=1$.

A. $\dfrac{\sqrt{21}}{10}$ B. $\dfrac{\sqrt{29}}{10}$ C. $\dfrac{\sqrt{21}}{5}$ D. $\dfrac{\sqrt{21}}{20}$ E. $\dfrac{\sqrt{29}}{5}$

34. Solve the logarithmic equation $\log_3 x+\log_9 x=6$ for x.

A. $\{18\}$ B. $\{27\}$ C. $\{36\}$ D. $\{81\}$ E. No solution

35. Solve the inequality $\dfrac{x^2+(a-b)x-ab}{x+c}\leq 0$ for x, assuming that $0<a<b<c$.

A. $(-\infty,-c)\cup(b,\infty]$ B. $(-c,-a)\cup(b,\infty)$ C. $(-\infty,-c)\cup(-a,b)$

D. $(-\infty,-c)\cup[-a,b]$ E. $(-\infty,-c]\cup(-a,b)$

36. How many different 4-letter words can be spelled using the letters from the word ALABAMA?

A. 24 B. 73 C. 35 D. 6 E. 840

37. Let $x=5\sec\theta$ and assume $0\leq\theta<\dfrac{\pi}{2}$. Simplify the expression $\dfrac{\sqrt{x^2-25}}{x}$.

A. $\cos\theta\tan\theta$ B. $\tan\theta$ C. $\sin\theta$ D. $5\sec\theta\tan\theta$ E. $\cos\theta$

38. Which of the following expressions is equivalent to $\dfrac{1-\cos 2\theta}{\sin 2\theta}$?

A. $\cot\theta$ B. $1-\sin 2\theta$ C. $\cos^2\theta$ D. $\tan\theta$ E. $\sin^2 2\theta$

39. One method of providing direction involves reference to a north-south line using an acute angle. Suppose an airplane flies true north at 550 miles per hour against a head wind blowing at $S40°E$ (that is, 40° east of south) at 75 miles per hour. What is the direction of the airplane?

A. $N84.7°E$ B. $N83.9°E$ C. $N97°E$ D. $N63.1°E$ E. $N85°E$

40. Solve the inequality $|x-1|-|x-4|\geq 7$ for x.

A. $(-\infty,1)$ B. $[1,4)$ C. $[4,\infty]$ D. $(-\infty,1]\cup[4,\infty)$ E. No solution