TENNESSEE MATHEMATICS TEACHERS ASSOCIATION

SIXTY-FIFTH ANNUAL MATHEMATICS CONTEST

2023

Precalculus

Scoring Formula: $4 \times$ (Number Right) $-$ (Number Wrong) $+ 40$

Directions:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

1. What is the value of $(3+2i)^4$ where $i=\sqrt{-1}$?

(a) $119+120i$ (b) $119-120i$ (c) $-119+120i$ (d) 97 (e) $-119-120i$

2. What is the solution set of inequality $\dfrac{x-1}{x+2} \geq \dfrac{2}{5}$?

(a) $(-\infty,-2)$
(b) $[3,\infty)$
(c) $(-2,\infty)$
(d) $(-\infty,-2)\cup[3,\infty)$
(e) $(-\infty,-2)\cup(-2,3]$

3. What is the domain of the function $f(x)=\tan^{-1}x$?

(a) $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ (b) $[-1,1]$ (c) $(-1,1)$ (d) $(-\infty,\infty)$ (e) $(0,\pi)$

4. Which of the following best describes the graph of the polar curve $r=5$?

(a) a vertical line (b) a horizontal line (c) a circle (d) a parabola (e) a spiral

5. Let $A=\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}$. What is the product AB?

(a) $\begin{bmatrix} -2 & 12 \\ 5 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 \\ -12 & 20 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 1 \\ -18 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 4 \\ -8 & 12 \end{bmatrix}$

(e) $\begin{bmatrix} 18 & -14 \\ 1 & 3 \end{bmatrix}$

6. What is the y-intercept of the line containing the points $(-1, 3)$ and $(5, 4)$?

(a) -19 (b) $\frac{6}{19}$ (c) -6 (d) $\frac{19}{6}$ (e) 4

7. A 7^{th} degree polynomial has at most how many extrema?

(a) 7 (b) 6 (c) 8 (d) 5 (e) 0

8. The half-life of Francium-223 is 22 minutes. To the nearest tenth of a mg, how much of a 100 mg sample will be left after an hour?

(a) 15.3 mg (b) 15.4 mg (c) 15.2 mg (d) 15.0 mg (e) 15.1 mg

9. What is the remainder when the polynomial $x^4 + 3x^2 - 5x + 1$ is divided by $x - 2$?

(a) 15 (b) 39 (c) 6 (d) 19 (e) 1

10. What is the solution to the equation $2^{2x-5} = 8^{x-1}$?

(a) $x = 2$ (b) $x = -1$ (c) $x = -2$ (d) $x = 0$ (e) $x = 1$

11. The master code on a burglar alarm consists of 5 digits (0-9). How many different codes are possible if consecutive digits in the code may not be identical?

(a) 15120
(b) 36864
(c) 100000
(d) 30240
(e) 65610

12. The points $(3, -12)$, $(-3, 3)$, and $\left(-4, -\frac{1}{3}\right)$ lie on the curve $y = ax^2 + bx + c$. What is the value of $a + b + c$?

(a) $-\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{6}$ (e) $-\frac{2}{3}$

13. What is the solution to the following linear system?

$$\begin{aligned} 2x - y + 3z &= 1 \\ x + 2y + z &= 2 \\ x + 3y - z &= 3 \end{aligned}$$

(a) $\left(\frac{2}{3}, \frac{11}{15}, \frac{2}{15}\right)$ (b) $(6.4, -0.2, -4)$ (c) $(-2, 1, 2)$ (d) $\left(\frac{8}{3}, \frac{1}{3}, -\frac{4}{3}\right)$ (e) $\left(\frac{10}{9}, \frac{5}{9}, -\frac{2}{9}\right)$

14. What is the sum of the series $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$?

(a) 3 (b) 2.75 (c) 4 (d) 2.25 (e) 2.5

15. The expression $\sec\theta - \sin\theta\tan\theta$ simplifies to which of the following?

(a) 0
(b) $\csc x$
(c) $\sin x$
(d) $\cos x$
(e) 1

16. What is the solution set to the inequality $\frac{(x^2-9)(x^2-2x+4)}{(x+1)(x-2)} \geq 0$?

(a) $(-\infty, -3] \cup (-1, \infty)$
(b) $[-3, -1) \cup [3, \infty)$
(c) $(-\infty, -3] \cup [-1, 2]$
(d) $(-\infty, -3] \cup (-1, 2) \cup [3, \infty)$
(e) $[-3, -1) \cup (2, \infty)$

17. The polynomial $2x^4 - 4x^3 + 11x^2 + 7x + c$ has $(x + 1)$ as a factor. What is the value of c?

(a) $c = 5$ (b) $c = -5$ (c) $c = -16$ (d) $c = 10$ (e) $c = -10$

18. What is the solution set to the equation $\log_6(x+5) + \log_6 x = 2$?

(a) $\{-4, 9\}$

(b) $\{4\}$

(c) $\{4, -9\}$

(d) $\left\{\dfrac{-5 \pm \sqrt{73}}{2}\right\}$

(e) $\left\{\dfrac{-5 + \sqrt{73}}{2}\right\}$

19. The point $\left(4, \dfrac{4\pi}{3}\right)$ in the polar coordinate system corresponds to which point in the Cartesian coordinate system?

(a) $\left(2, 2\sqrt{3}\right)$

(b) $\left(2\sqrt{3}, 2\right)$

(c) $\left(-2\sqrt{2}, -2\sqrt{2}\right)$

(d) $\left(-2\sqrt{3}, -2\right)$

(e) $\left(-2, -2\sqrt{3}\right)$

20. What is the inverse of the function $f(x) = \dfrac{2x-1}{3x+5}$?

(a) $f^{-1}(x) = \dfrac{3x-1}{5x-2}$

(b) $f^{-1}(x) = \dfrac{2x+1}{3x-5}$

(c) $f^{-1}(x) = \dfrac{3x+5}{2x-1}$

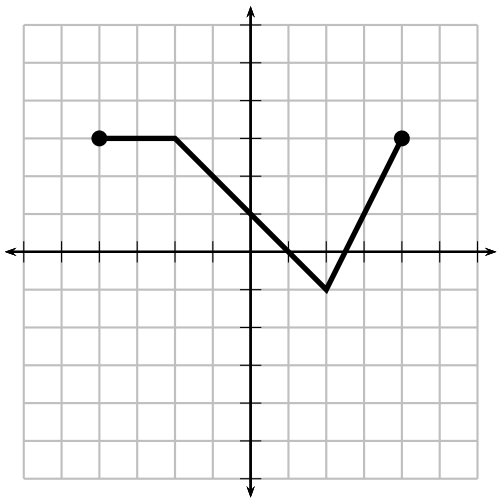
(d) $f^{-1}(x) = \dfrac{5x+1}{2-3x}$

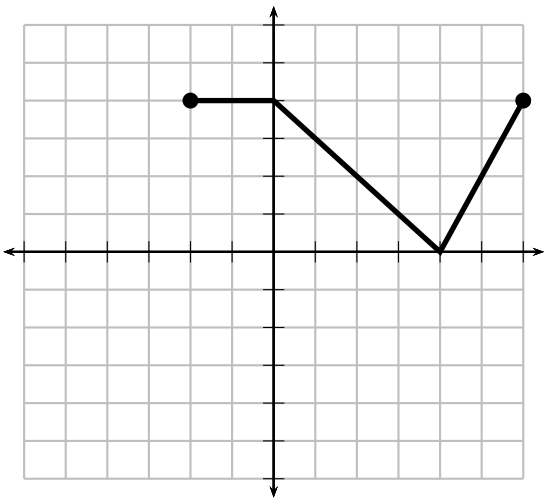
(e) $f^{-1}(x) = \dfrac{1-3x}{5x-2}$

21. An arithmetic sequence with 17 terms has a sum of 2023. If the last term is 207, what is the first term?

(a) 35 (b) 29 (c) 33 (d) 31 (e) 37

22. The graph on the left below is for $y = f(x)$. Which of the following describes the graph on the right?





(a) $y = f(x-2)+1$
(b) $y = f(x-2)-1$
(c) $y = f(x+2)+1$
(d) $y = f(x+2)-1$
(e) $y = f(2x)+1$

23. What is the inverse of the function $f(x) = \dfrac{e^x - e^{-x}}{2}$?

(a) $f^{-1}(x) = \ln(x + \sqrt{x^2+1})$
(b) $f^{-1}(x) = \ln(x + \sqrt{x^2-1})$
(c) $f^{-1}(x) = \ln(x - \sqrt{x^2-1})$
(d) $f^{-1}(x) = \ln(\sqrt{x^2+1} - x)$
(e) $f^{-1}(x) = \dfrac{2}{e^x - e^{-x}}$

24. Let $f(x) = 1 - \sqrt{x}$. What is the domain of $f(f(x))$?

(a) $[0, \infty)$ (b) $[-1, 1]$ (c) $[1, \infty)$ (d) $[0, 1]$ (e) $\emptyset$

25. How many solutions to the equation $2\sin^2(3x) - 1 = 0$ on the interval $[0, \pi]$?

(a) 4 (b) 6 (c) 2 (d) 0 (e) 1

26. What is the angle between the vectors $(1, -4)$ and $(7, 2)$ rounded to the nearest tenth of a degree?

(a) $90.0°$ (b) $91.9°$ (c) $1.9°$ (d) $88.1°$ (e) $178.1°$

27. What is the solution set for the equation $\sqrt{4 - x} = 2x + 13$?

(a) $\left\{5, \frac{35}{4}\right\}$ (b) $\emptyset$ (c) $\left\{-\frac{35}{4}\right\}$ (d) $\left\{-5, -\frac{35}{4}\right\}$ (e) $\{-5\}$

28. What is the exact area of the triangle with sides of length 5 units, 7 units, and 10 units?

(a) $5\sqrt{14}$ square units

(b) $\frac{5\sqrt{70}}{2}$ square units

(c) 17.5 square units

(d) $2\sqrt{66}$ square units

(e) 25 square units

29. What are all the complex roots of the polynomial equation: $x^4 - 3x^3 - 3x^2 - 3x - 4 = 0$?

(a) $-1, 4, \pm 2i$ (b) $-1, 4, \pm i$ (c) $\pm 1, \pm 2$ (d) $-4, 1, \pm 2i$ (e) $-4, 1, \pm i$

30. If $\sin x = \frac{3}{4}$ and $\sin y = \frac{1}{5}$ where both x and y are in quadrant I, what is the value of $\sin(x - y)$?

(a) $\frac{6\sqrt{3} - \sqrt{7}}{20}$ (b) $\frac{6\sqrt{6} - \sqrt{7}}{20}$ (c) $\frac{11}{20}$ (d) $\frac{6\sqrt{6} + \sqrt{7}}{20}$ (e) $\frac{6\sqrt{3} - \sqrt{7}}{20}$

31. A triangle has sides of length 7 cm, 10 cm, and 12 cm. What is the measure of the smallest angle to the nearest degree?

(a) $1°$ (b) $36°$ (c) $88°$ (d) $56°$ (e) $52°$

32. A company the manufactures metal cans is designing a cylindrical can that has height 6 inches and diameter 3 inches. If the top costs \$0.13 per square inch, the side costs \$0.07 per square inch, and the bottom costs \$0.22 per square inch, how much would the material for one can cost rounded to the nearest cent?

(a) \$6.43 (b) \$17.81 (c) \$10.88 (d) \$4.45 (e) \$13.85

33. A precalculus class consists of 8 juniors and 10 seniors. If 4 students from the class are randomly selected to compete in the TMTA High School Mathematics Contest, what is the probability that 2 juniors and 2 seniors are chosen?

(a) $\frac{2}{5}$ (b) $\frac{7}{17}$ (c) $\frac{40}{153}$ (d) $\frac{20}{81}$ (e) $\frac{7}{102}$

34. Let $A = \begin{bmatrix} 3 & -x \\ x & 4 \end{bmatrix}$. What is A^{-1}?

(a) $\begin{bmatrix} \frac{1}{3} & -\frac{1}{x} \\ \frac{1}{x} & \frac{1}{4} \end{bmatrix}$

(b) $\frac{1}{12+x^2}\begin{bmatrix} 3 & -x \\ x & 4 \end{bmatrix}$

(c) $\frac{1}{12+x^2}\begin{bmatrix} -3 & x \\ -x & -4 \end{bmatrix}$

(d) $\frac{1}{12-x^2}\begin{bmatrix} 4 & x \\ -x & 3 \end{bmatrix}$

(e) $\frac{1}{12+x^2}\begin{bmatrix} 4 & x \\ -x & 3 \end{bmatrix}$

35. What are the foci of the ellipse with the equation $x^2 + 6y^2 - 6x + 36y + 57 = 0$?

(a) $(3-\sqrt{2}, -3)$ and $(3+\sqrt{2}, -3)$

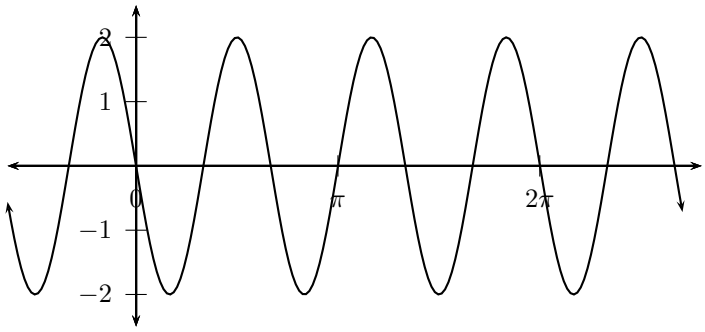
(b) $(2, -3)$ and $(4, -3)$

(c) $(3-\sqrt{5}, -3)$ and $(3+\sqrt{5}, -3)$

(d) $(1, -3)$ and $(5, -3)$

(e) $(3-\sqrt{3}, -3)$ and $(3+\sqrt{3}, -3)$

36. Which of the following is an equation for the graph shown below?



(a) $y = 2\sin\left(\frac{x}{3}\right)$
(b) $y = -2\sin\left(\frac{x}{3}\right)$
(c) $y = -2\sin(3x)$
(d) $y = -2\cos(3x)$
(e) $y = 2\cos(3x)$

37. For independent events A and B where $P(A) = 0.3$ and $P(B) = 0.6$, what is the $P(A \text{ or } B)$?

(a) 0.9 (b) 0.3 (c) 0 (d) 0.72 (e) 0.18

38. If $\cos\theta = \frac{4}{5}$, what is $\cos(2\theta)$?

(a) $\frac{16}{25}$ (b) $\frac{9}{25}$ (c) $\frac{7}{25}$ (d) $\frac{2}{5}$ (e) $\frac{12}{25}$

39. What is the positive value of b such that the line $y = 2x + b$ and the circle $x^2 + y^2 = r^2$ (where $r > 0$) intersect in exactly one point?

(a) $b = 2r$ (b) $b = \sqrt{5}r$ (c) $b = r + 2$ (d) $b = \sqrt{2}r$ (e) $b = \sqrt{3}r$

40. What is the distance from the origin to the line with equation $5x + 6y = 30$?

(a) $\frac{32\sqrt{61}}{61}$ (b) $\frac{12\sqrt{11}}{11}$ (c) $\frac{15\sqrt{11}}{11}$ (d) $\frac{30\sqrt{61}}{61}$ (e) $\frac{36\sqrt{61}}{61}$