TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

SIXTIETH ANNUAL MATHEMATICS CONTEST

2016

Precalculus

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Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the <u>best</u> answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase <u>completely</u>. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

1. What is the value of $\dfrac{\ln\left(8^{\log_2(4)}\right)}{\ln\left(\frac{\log 4}{\log 2}\right)}$?

   A. 4 B. 3 C. -4 D. $\ln 4$ E. none of the above

2. Which of the following describes the sum of the coefficients in $(x^8 + 5x - 4)^{18}$ when this polynomial is fully expanded?

   A. The sum is less than 25000.

   B. The sum is 25412.

   C. The sum is 35200.

   D. The sum is 262144.

   E. The sum is 531441.

3. Consider the series $\frac{-1}{3} + 1 - 3 + 9 - 27$. Which of the following is a correct representation of this series?

   A. $\sum_{n=1}^{5} 3^n$ B. $\sum_{n=1}^{5} 3^{-n}$ C. $\sum_{n=2}^{6}(-1)^{n-1}3^n$ D. $\sum_{n=1}^{5}(-1)^n 3^{n-2}$ E. $\sum_{n=2}^{6}(-1)^n 3^{n-2}$

4. Which of the following is the expansion of $(3a - b)^5$?

   A. $243a^5 - b^5 4$

   B. $243a^5 - 6ab - b^5$

   C. $243a^5 - 324a^4b + 162a^3b^2 - 36a^2b^3 + 3ab^4 - b^5$

   D. $243a^5 - 81a^4b + 27a^3b^2 - 9a^2b^3 + 3ab^4 - b^5$

   E. $243a^5 - 405a^4b + 270a^3b^2 - 90a^2b^3 + 15ab^4 - b^5$

5. What is the domain of the function $\cos^{-1}(\tan(\theta))$ with θ measured in radians?

   A. $[-\pi/4,\ \pi/4]$

   B. the union of $[n\pi/4,\ (n{+}1)\pi/4]$ for each integer n

   C. the union of $[(2n{-}1)\pi/4,\ (2n)\pi/4]$ for each integer n

   D. the union of $[(4n{-}1)\pi/4,\ (4n{+}1)\pi/4]$ for each integer n

   E. the union of $[2n\pi/4,\ (2n{+}1)\pi/4]$ for each integer n

6. Suppose $\cos(\alpha) = 0.7$. Which of the following is NOT true?

   A. $\cos(-\alpha) = 0.7$

   B. $\cos(\pi + \alpha) = 0.7$

   C. $\cos(\pi - \alpha) = -0.7$

   D. $|\sin(\alpha)| = \sqrt{0.51}$

   E. $\cos(2\pi + \alpha) = 0.7$

7. Jacob is filling water balloons. He uses a hose that releases 500 cubic centimeters of water each second. As a balloon fills with water, its approximate surface area A is related to its volume V by the equation $A = 4.8V^{2/3}$. If we let t represent time in seconds, which of the following is a formula for the approximate surface area (in cm^2) as a function of time?

   A. $A = 4.8(500t)^{2/3}$

   B. $A = 4.8\left(\frac{t}{500}\right)^{2/3}$

   C. $A = \left(\frac{4.8t}{500}\right)^{2/3}$

   D. $A = 4.8(500t)^{3/2}$

   E. $A = 4.8\left(\frac{t}{500}\right)^{3/2}$

8. While taking this test Hunter makes twice as many mistakes as Jesse and half as many as Jamie. Together they make forty-two mistakes. How many mistakes did Jamie make?

   A. 6 B. 12 C. 16 D. 24 E. 30

9. Harper has two similar triangles. The smaller has an area of 56 square units. The larger has a perimeter which is six times the smaller. The area of the larger triangle is?

   A. 62 B. 336 C. 1440 D. 2016 E. 2880

10. Suppose θ is an acute angle and $\sin\theta = x$. Which of the following is $\cot\theta$?

   A. $\dfrac{x}{\sqrt{1-x^2}}$ B. $\sqrt{1-x^2}$ C. $\dfrac{\sqrt{1-x^2}}{x}$ D. $\frac{x}{1-x^2}$ E. $\dfrac{1-x^2}{x}$

11. Let a, b, c, and d be complex numbers and let $p(z) = az^4 + bz^3 + cz^2 + d$. Suppose $p(z)$ is zero for $z = 1$, $z = 2$, $z = 3+i$, $z = 5-i$ and $z = 11$. Which of the following is NOT true for all complex numbers z?

   A. $p(z)^3 \geq p(z)^2$

   B. $a^2 + b^2 + c^2 + d^2 = 0$

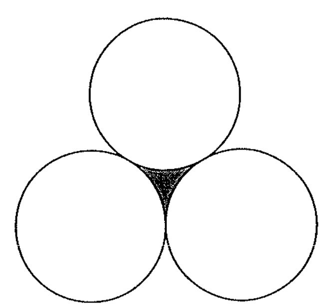
   C. $az^2 + b = c$

   D. $-|p(z)| < p(z)^2$

   E. $a^4z + b^3z + c^2z + d = 0$

12. The tangent circles in the drawing on the right all have a radius of one unit. What is the area of the shaded region?

   A. $4-\pi$ B. 2π C. $\pi/4$ D. $\sqrt{3}-\frac{\pi}{2}$ E. $2\sqrt{3}-\pi$



13. Which of the following pairs of functions do NOT have the same domain?

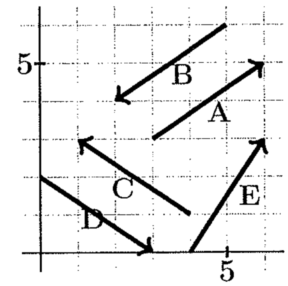
A. $\cos x$ and $\sin x$

B. $\tan x$ and $\cot(x + \pi/2)$

C. $\ln x$ and $\sqrt{1/x}$

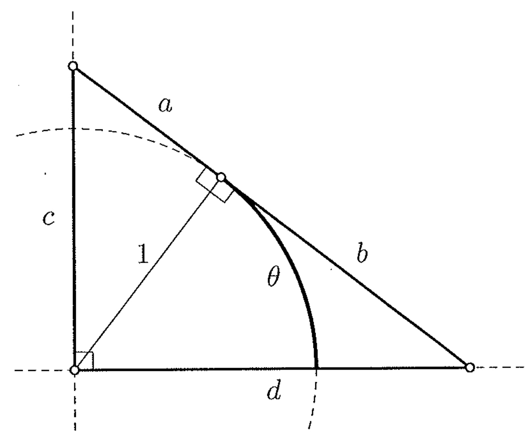
D. $\sec x$ and $\csc x$

E. e^x and $\sin x$



14. Let **V** be the vector with horizontal component 4 and vertical component -5. Which of the following vectors has a dot product with **V** of -22?

A. **A** B. **B** C. **C** D. **D** E. **E**



15. An arc length θ is measured on the unit circle and a tangent line is drawn where the arc terminates. Which of the indicated segments has length $\csc\theta$?

A. a B. b C. c D. d E. $a+b$

16. Use the exact value of $\cos\frac{\pi}{6}$ and the half-angle identity to find $\cos\frac{\pi}{12}$.

A. $\frac{\sqrt{338-144\sqrt{2}}}{12}$

B. $\ln\left(\frac{444}{169}\right)$

C. $\sqrt{1-\frac{14}{209}}$

D. $\frac{4}{\sqrt{7\sqrt{6}}}$

E. $\frac{\sqrt{2+\sqrt{3}}}{2}$

17. Find the determinant $\begin{vmatrix} a & a & 1 \\ a & a+c & b \\ a & a+c & d \end{vmatrix}$.

A. 0 B. $b^2 - 4ac$ C. $a^3(a+c)^2 d$ D. $ac(d-b)$ E. $ac^2(b+d)^3$

18. How many right triangles are there with integer length sides a, b and hypotenuse $b + 1$ where none of these lengths is greater than 2015^2?

A. less than 403

B. between 403 and 805

C. between 806 and 1208

D. between 1209 and 1611

E. between 1612 and 2015

19. The function $y = \sin x + \cos x$ can be written as $y = d\sin(x + \frac{\pi}{4})$ where d has the value

A. $\sqrt{2}$ B. $\sqrt{2}/2$ C. 2 D. 1/2 E. 3/2

20. A comet travels according to the parametric equations $x = 5 - (t-1)^2$ and $y = 1 - (t-3)^2$ where time $t \geq 0$ is measured in hours. This path has been determined by placing Earth at the origin. Determine at what time (in hours) the comet is closest to Earth. Round your answer to the nearest hundredth of an hour.

A. 1.25 B. 2.84 C. 3.19 D. 3.26 E. 3.27

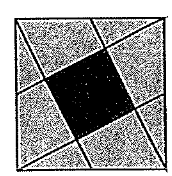
21. On the Richter scale, the magnitude M of an earthquake is related to its intensity I by the equation $M = log(\frac{I}{I_0})$, where I_0 is a constant reference intensity. How many times more intense is an earthquake of magnitude 7.3 than an earthquake of magnitude 5.7? Round your answer to the nearest tenth.

A. 1.3 B. 3.1 C. 5.0 D. 18.7 E. 39.8

22. Radon-222 is a radioactive gas with a half-life of approximately 3.82 days. Suppose an 11 picogram sample of radon-222 is isolated. How many picograms of of radon-222 will remain after 11 days? Round your answer to the nearest tenth of a picogram.

A. 1.0 B. 1.4 C. 1.5 D. 1.7 E. 2.0

23. Bisect the sides of a square, and draw "diagonals" as indicated on the right. The length of a side of the small dark square is what proportion of the length of a side of the original square?



A. $\sqrt{5}/5$ B. 1/4 C. 1/2 D. $\sqrt{2}/2$ E. $\sqrt{3}/3$

24. What is the sum of $\sin(\pi/2) + \sin(2\pi/2) + \sin(3\pi/2) + \ldots + \sin(2016\pi/2)$.

A. -1 B. 0 C. 1 D. 2 E. 3

25. Consider the set of all intervals of the form $\left(\frac{n}{m^2+n^2+1}, \frac{m}{m^2+n^2+1}\right)$ where $m - n \geq \frac{1}{12}$. If one of these intervals is chosen at random, what is the probability the interval contains both a rational number and an irrational number?

A. $\frac{1}{(12^2+1)}$ B. $\frac{1}{12^2}$ C. $\frac{1}{13}$ D. $\frac{1}{12}$ E. 1

26. The graph of $x = 6789t^2 - 2015t^4$ and $y = -t^2 - 3^{2016}$ is

A. a parabola that opens to the left

B. a parabola that opens to the right

C. a parabola that opens downward

D. a parabola that opens upward

E. an ellipse

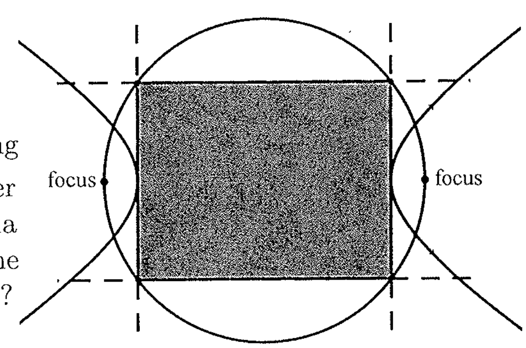
27. The sine function is not one-to-one, so to define an inverse sine function, we typically restrict the domain of the sine function to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. This choice of interval is somewhat arbitrary. Suppose we define $y = SIN(x)$ to mean that $y = sin(x)$ and $\frac{-3\pi}{2} \leq x \leq \frac{-\pi}{2}$. With this definition, what is $SIN^{-1}(\frac{-1}{2})$?

A. $\frac{-\pi}{6}$ B. $\frac{-\pi}{3}$ C. $\frac{-2\pi}{3}$ D. $\frac{-4\pi}{3}$ E. $\frac{-5\pi}{6}$

28. Find the standard form of the equation of a hyperbola (centered at the origin) with vertices $(\pm 10, 0)$ and foci $(\pm 26, 0)$.

A. $\frac{x^2}{81} - \frac{y^2}{144} = 1$ B. $\frac{y^2}{81} - \frac{x^2}{144} = 1$ C. $\frac{y^2}{576} - \frac{x^2}{100} = 1$ D. $\frac{x^2}{100} - \frac{y^2}{576} = 1$ E. $\frac{x^2}{100} - \frac{y^2}{144} = 1$

29. Given a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, draw the circle using the segment connecting the two foci as a diameter Draw vertical lines at the vertices of the hyperbola Draw horizontal lines where these lines intersect the circle. What is the area of the resulting rectangle?



A. $2|a+b|$ B. $2|ab|$ C. $a^2 + b^2$ D. $\sqrt{a^2+b^2}$ E. $4|ab|$

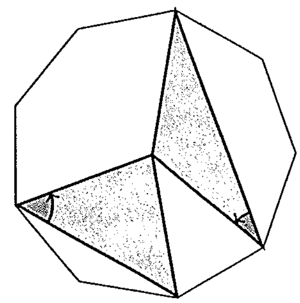
30. How many solutions in real numbers x, y, z does the following equation have?

$$\begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} y & 1 & 0 \\ 1 & y & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A. 3 B. 4 C. 5 D. 6 E. 8

31. Two triangles are formed in a regular nonagon by connecting the center to pairs of vertices (see the picture on the right). What is the sum of the two indicated angles in radians?

A. $4\pi/9$ B. $5\pi/9$ C. $2\pi/3$ D. $7\pi/9$ E. $8\pi/9$



32. If $2 = x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}}$ (an infinite tower of exponents), then x is

A. $2^{1/2}$ B. $e^{1/2}$ C. $1/e^2$ D. $2/e$ E. $\ln 2$

33. If $\sin\alpha + \sin\beta = A$ and $\cos\alpha + \cos\beta = B$, then what is $2\cos(\alpha - \beta)$?

A. $2AB$ B. $A^2 + B^2 - 2$ C. $2A^2 + 2B^2 + 2$ D. $A^2 - B^2 + 2$ E. $A^2 - B^2 + 1$

34. Two unit squares (each side has length one unit) are drawn so that the corner of the second square is drawn through the center of the first square. If the second square is rotated an angle of θ radians with respect to the first square, what is the area of the region where the interiors of the two squares overlap?

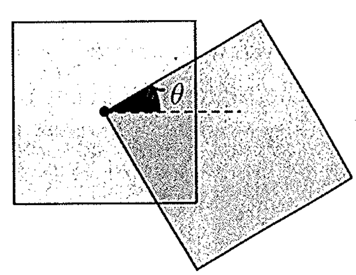
A. $(\sec^2\theta - \tan^2\theta)/4$

B. $(\sin^2\theta - \cos^2\theta)/4$

C. $(\sin\theta\cos\theta)/2$

D. $(\sec^2\theta + \tan^2\theta)/4$

E. $(\sin\theta\cos\theta)/4$



35. A square inscribed in a circle of radius 1 has an area of 2. There is only one other regular n-gon ($n \geq 3$) which when inscribed in a unit circle has an integer area. Which is it?

A. pentagon ($n = 5$)

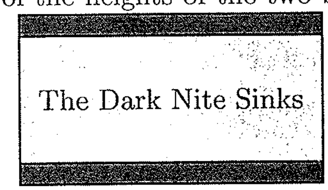
B. hexagon ($n = 6$)

C. nonagon ($n = 9$)

D. decagon ($n = 10$)

E. dodecagon ($n = 12$)

36. High-definition television screens have an aspect ratio (ratio of width to height) of 16:9. Wide-screen films (aspect ratio 2.39:1) are shown on such television sets using the full width of the screen, leaving black bars on the top and bottom. If the diagonal of the screen measures 60 inches, what is the sum, in inches, of the heights of the two black bars? Round your answer to the nearest thousandth of an inch.

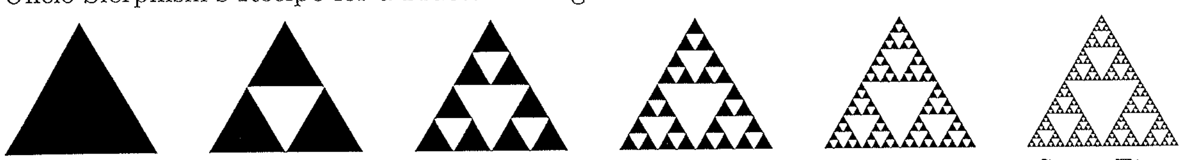


A. 2.535 B. 3.768 C. 5.533 D. 7.535 E. 8.645

37. The graph of which of the following polar equations is NOT an ellipse?

A. $r = \dfrac{2}{4 - \cos\theta}$ B. $r = \dfrac{-3}{6 - 2\sin\theta}$ C. $r = \dfrac{-4}{3 - 2\cos\theta}$ D. $r = \dfrac{5}{5 - 4\sin\theta}$ E. $r = \dfrac{7}{4 - 4\cos\theta}$

38. Uncle Sierpiński's Recipe for a Fractal Triangle:



Stage Zero Stage One Stage Two Stage Three Stage Four Stage Five

Start at "stage zero" with a solid (filled) equilateral triangle with area one square unit. To find each subsequent stage, subdivide each solid triangle into four smaller congruent equilateral triangles and remove the central one. What is the total area of the solid triangles in stage twenty?

A. $1 - \sum_{n=0}^{19} \left(\frac{3^n}{4}\right)$

B. $1 - \sum_{n=0}^{19} \left(\frac{3}{4}\right)^n$

C. $1 - \sum_{n=1}^{20} \left(\frac{3}{4}\right)^n$

D. $1 - \sum_{n=0}^{19} \left(\frac{3^{n-1}}{4^n}\right)$

E. $1 - \frac{1}{3} \sum_{n=1}^{20} \left(\frac{3}{4}\right)^n$

39. An object dropped from a plane at 500 meters above ground level has velocity given by $v(t) = 49(1 - e^{-t/5})$ and position above ground level given by $s(t) = 745 - 49(t + 5e^{-t/5})$. Time t is measured in seconds, velocity is measured in meters per second, and position is measured in meters. At what velocity will the object hit the ground? Round your answer to the nearest hundredth.

A. 32.89 meters per second

B. 46.40 meters per second

C. 46.54 meters per second

D. 46.56 meters per second

E. 49.00 meters per second

40. Evaluate the sum $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + \sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}} + \cdots + \sqrt{1+\frac{1}{2015^2}+\frac{1}{2016^2}}$.

A. $\frac{2016^2}{2015}$ B. $\frac{2014 \cdot 2016}{2015^2}$ C. $\frac{2016 \cdot 2018}{2017^2}$ D. $2015 - \frac{1}{2015}$ E. $2016 - \frac{1}{2016}$