FIFTIETH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Precalculus 2006

Prepared by:

Department of Mathematics and Computer Science
University of the South
Sewanee, Tennessee

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, Tennessee

Coordinated by: Catherine Cavagnaro

Scoring formula: $4R - W + 40$

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State Community College, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
TRW Commercial Steering Division, Lebanon, Tennessee
Wright Industries, Inc., Nashville, Tennessee

1. Solve: $3e^x - 1 = e^x$

   A. $x = \ln(2)$
   B. $x = -\ln(2)$
   C. $x = \ln(3)$
   D. $x = -\ln(3)$
   E. No solution.

2. Which of the following equations is *not* an identity?

   A. $\sin^2(x) + \cos^2(x) = 1$
   B. $\cos(2x) = \cos^2(x) - \sin^2(x)$
   C. $\cos(2x) = 1 - 2\sin^2(x)$
   D. $\cos(2x) = 2\cos^2(x) - 1$
   E. $\sin(2x) = 2\sin(x) + \cos(x)$

3. If $h(x) = \sqrt[3]{\frac{3}{x+2}}$ and $h = f \circ g$, which of the following are possibilities for f and g?

   A. $f(x) = \sqrt[3]{x}$ and $g(x) = x + 2$
   B. $f(x) = \sqrt[3]{\frac{3}{x}}$ and $g(x) = x + 2$
   C. $f(x) = \sqrt[3]{\frac{3}{x}}$ and $g(x) = \frac{3}{x+2}$
   D. $f(x) = \frac{3}{x+2}$ and $g(x) = \sqrt[3]{x}$
   E. $f(x) = x + 2$ and $g(x) = \sqrt[3]{\frac{3}{x}}$

4. If a, b, c, and d are nonzero constants, then the graph of $y = \frac{ax+b}{cx+d}$ has a horizontal asymptote at:

   A. $y = \frac{a}{c}$
   B. $y = \frac{b}{d}$
   C. $y = -\frac{d}{c}$
   D. $y = \frac{d}{c}$
   E. The graph has no horizontal asymptote.

5. If $m(x) = (x^3 + 1)^5$, then $m^{-1}(x) =$

   A. $\sqrt[15]{x} - 1$
   B. $\sqrt[3]{\sqrt[5]{x} - 1}$
   C. $\sqrt[5]{\sqrt[3]{x} - 1}$
   D. $\sqrt[5]{\sqrt[3]{x} + 1}$
   E. m has no inverse function.

6. For what values of x is $x^2 - x < 6$?

   A. $x \in (-\infty, -2) \cup (3, \infty)$
   B. $x \in (-\infty, 0) \cup (6, \infty)$
   C. $x \in (-\infty, -6) \cup (0, \infty)$
   D. $x \in (-2, 3)$
   E. $x \in (0, 6)$

7. If $\log(w) = \log(x) - 2\log(y) - \log(z)$, then $w =$

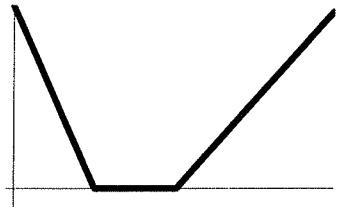
   A. $\frac{x}{2yz}$
   B. $\frac{xz}{2y}$
   C. $\frac{xz}{y^2}$
   D. $\frac{x}{y^2 z}$
   E. $\frac{2x}{yz}$

8. Find the solution set to the equation: $\log_2(x^2) = 6$

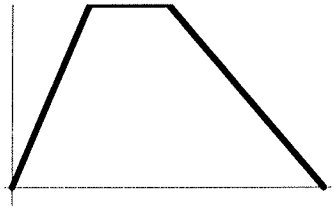
   A. $\{6\}$
   B. $\{8\}$
   C. $\{-6, 6\}$
   D. $\{-8, 8\}$
   E. $\{6, 8\}$

9. You walk to the store quickly, stay there for a minute, and walk back home more slowly. Which graph best represents your distance from the store as a function of time?

A.



B.



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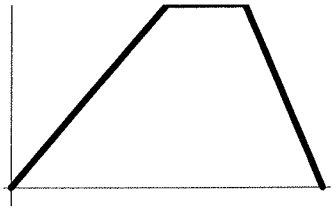

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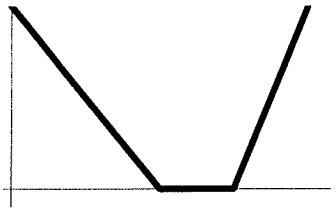

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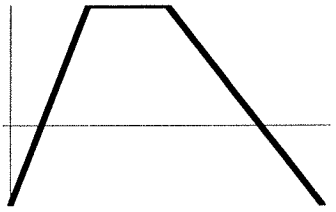


10. Which of the following expressions is equivalent to $\cos(\sin^{-1}(x))$?

A. $\sqrt{1-x^2}$
B. $-\sqrt{1-x^2}$
C. $\frac{x}{\sqrt{1-x^2}}$
D. $-\frac{x}{\sqrt{1-x^2}}$
E. $\frac{\sqrt{1-x^2}}{x}$

11. Which function's domain is $(-\infty, 5]$?

A. $f(x) = \sqrt{5-x}$
B. $f(x) = \sqrt{x-5}$
C. $f(x) = \frac{1}{\sqrt{5-x}}$
D. $f(x) = \frac{1}{\sqrt{x-5}}$
E. $f(x) = \frac{x-5}{\sqrt{x-5}}$

12. Which function's range is all real numbers?

A. $f(x) = \frac{1}{x}$
B. $f(x) = e^x$
C. $f(x) = \ln(x)$
D. $f(x) = \sin(x)$
E. $f(x) = \tan^{-1}(x)$

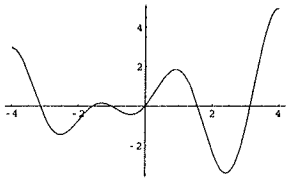
13. Write the surface area of a cube in terms of its volume.

A. $S = 4\sqrt{V}$
B. $S = \frac{V^2}{16}$
C. $S = 6V^{2/3}$
D. $S = \frac{V^{3/2}}{6}$
E. None of these

14. Find the solution set of the inequality: $|2x - 7| < 3$.

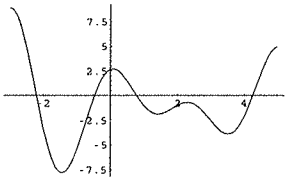
A. $(2, 5)$
B. $[2, 5]$
C. $(-\infty, 5)$
D. $(-\infty, 5]$
E. $(-\infty, \infty)$

15. Shown below is the graph of a function f.



If $g(t) = 2f(1-t) - 1$, which is the graph of g?

A.



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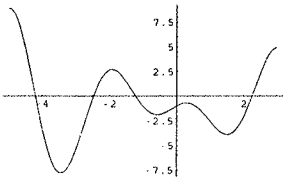

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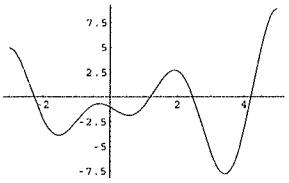

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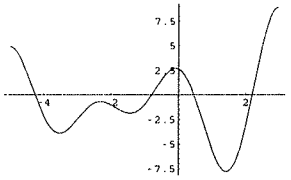

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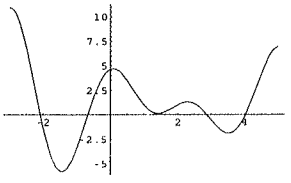


16. If $f(x) = kb^x$, $f(1) = 3$, and $f(2) = 5$, what are k and b?

A. $k = \frac{5}{3}$, $b = \frac{9}{5}$
B. $k = \frac{1}{3}$, $b = \frac{5}{9}$
C. $k = \frac{9}{5}$, $b = \frac{5}{3}$
D. $k = \frac{5}{9}$, $b = \frac{3}{5}$
E. $k = \frac{1}{9}$, $b = \frac{5}{3}$

17. Find the equation of a line passing through the point $(2, -5)$ and perpendicular to the line $x + 2y = 1$.

A. $y = 2x - 9$
B. $y = -2x - 1$
C. $y = \frac{1}{2}x - 6$
D. $y = -\frac{1}{2}x - 4$
E. $y = x - 7$

18. If $y = \frac{x^2}{x^{-4}}$ and $x > 0$, then $\frac{y}{\sqrt{x}} =$

A. x^3
B. $x^{5/2}$
C. $-x^{7/2}$
D. $\sqrt{x^9}$
E. $\sqrt{x^{11}}$

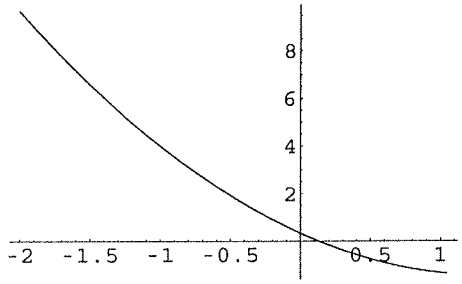
19. Let $k(x) = \frac{1}{x}$. Compute and simplify $\frac{k(2+h)-k(2)}{h}$.

A. $-\frac{1}{4+2h}$
B. $\frac{1}{h^2}$
C. $-\frac{1}{h^2}$
D. 1
E. $\frac{1}{2}$

20. Simplify $\frac{1}{x-1} - \frac{x-3}{(x-1)^2} + \frac{x+2}{(x-1)^3}$.

A. $\frac{3x}{(x-1)^3}$
B. $\frac{2}{(x-1)^3}$
C. $\frac{2}{(x-1)^6}$
D. $\frac{3x-2}{(x-1)^3}$
E. $\frac{3x-2}{(x-1)^6}$

21. Shown is the graph of a function g. Which number is closest to $g^{-1}(4)$?



A. -2

B. -1

C. 0

D. 1

E. 2

22. Shown are tables for functions f and g.

x	1	2	3	4	5
$f(x)$	2	4	1	5	3

x	1	2	3	4	5
$g(x)$	5	4	3	2	1

What is $(g \circ f)(1) - (f \circ g)(2)$?

A. -3

B. -1

C. 1

D. 3

E. 5

23. If $\sin(\theta) = -\frac{1}{3}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, what is $\tan(\theta)$?

A. $\frac{2\sqrt{2}}{9}$

B. $-\frac{2\sqrt{2}}{9}$

C. $\frac{\sqrt{2}}{4}$

D. $-\frac{\sqrt{2}}{4}$

E. None of these

24. Find all solutions to the equation: $\sin^2(x) = \sin(x)$.

A. $\{k\pi \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\}$

B. $\{\frac{(2k+1)\pi}{2} \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\}$

C. $\{\frac{(4k+1)\pi}{2} \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\}$

D. $\{k\pi \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\} \cup \{\frac{(4k+1)\pi}{2} \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\}$

E. $\{k\pi \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\} \cup \{\frac{(2k+1)\pi}{2} \mid k = \cdots, -2, -1, 0, 1, 2, \cdots\}$

25. A rectangle is drawn in the xy-plane so that its bottom edge lies on the x-axis with both bottom corners between $x = 0$ and $x = \pi$. Its top corners lie on the graph of $y = \sin(x)$. What is the maximum area of such a rectangle, accurate to two decimal places?

A. .71

B. .89

C. 1.01

D. 1.12

E. 1.27

26. What is the maximum value of the function $g(t) = \frac{\sin(1/t)}{t}$?

A. 1

B. $\sqrt{2}$

C. $2\sqrt{2}$

D. $\frac{1}{\sqrt{2}}$

E. $g(t)$ has no maximum value.

27. If the domain of f is $[0, \frac{\pi}{2}]$ and $f(x) = 1 + 2^x - \cos(x)$, what is $f^{-1}(1)$?

A. -1
B. 0
C. 1
D. 2
E. undefined

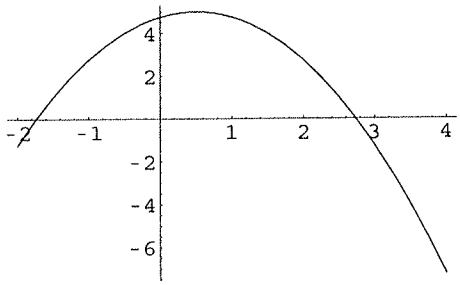
28. If k is a nonnegative integer, how many roots does $\sin(kx)$ have in the interval $[0, k\pi]$?

A. $2k$
B. k
C. k^2
D. $k^2 + 1$
E. $k + 1$

29. For what value(s) of a does the function $f(x) = x^3 - x^2 + ax - a$ have exactly two roots?

A. $a = -1$
B. $a = 0$
C. $a = 1$
D. $a \in \{-1, 0\}$
E. $a \in \{0, 1\}$

30. Shown is the graph of a function j. Which number is closest to $\frac{j(2.01)-j(2)}{.01}$?



A. -6
B. -3
C. 0
D. 3
E. 6

31. Two boats leave the same dock at the same time. One travels due north at a constant speed of 20 miles per hour. The other travels southeast at a constant speed of 10 miles per hour. Define a function d by

$d(t)$ = the distance (in miles) between the two boats t hours after they left the dock.

Find a formula for $d(t)$.

A. $d(t) = (500 + 200\sqrt{2})t^2$
B. $d(t) = (500 - 200\sqrt{2})t^2$
C. $d(t) = \sqrt{500 + 200\sqrt{2}}\ t$
D. $d(t) = \sqrt{500}\ t$
E. $d(t) = \sqrt{500t}$

32. A circle of radius three is formed by a piece of wire. If you take that same piece of wire and bend it into the shape of an equilateral triangle, what will be the area of that triangle?

A. $\frac{2\pi}{3}$
B. 6π
C. 9π
D. $\frac{9\sqrt{3}\pi^2}{2}$
E. $\sqrt{3}\ \pi^2$

33. If $\sum_{j=1}^{k}(2j + 10) = 25k$, then what is k?

A. 8
B. 11
C. 14
D. 17
E. 20

34. If a and b are real numbers, what is the real part of the complex number $\frac{i}{a+bi}$?

A. $\frac{1}{b}$
B. b
C. $-\frac{a}{b}$
D. $\frac{a}{b^2}$
E. $\frac{b}{a^2+b^2}$

35. If two fair six-sided dice are rolled, what is the probability that the product of the two dice is greater than 11?

A. $\frac{1}{4}$
B. $\frac{11}{36}$
C. $\frac{7}{18}$
D. $\frac{17}{36}$
E. $\frac{5}{9}$

36. Suppose that your school has three physics teachers and four math teachers. From this group of seven people, a committee of three is to be formed. The only restriction is that at least one of the three must be a physics teacher. How many different committees are possible?

A. 21
B. 31
C. 35
D. 45
E. 210

37. For what value of α (if any) does the following system of equations have infinitely many solutions?

$$\begin{aligned} 3x + 2y &= 0 \\ x + \alpha y &= 0 \end{aligned}$$

A. 3
B. 2
C. $\frac{2}{3}$
D. $\frac{3}{2}$
E. There is no value of α for which that system has infinitely many solutions.

38. Consider this set of polar coordinates: $\{(r, \theta) \mid r = 1, -\pi \leq \theta \leq 0\}$. The shape of the figure defined by that set can best be described as:

A. A square
B. A line segment
C. A circle
D. A semicircle
E. An ellipse

39. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, what is $2A^2 + A$?

A. $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 6 \\ 15 & 28 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 10 \\ 21 & 36 \end{bmatrix}$

D. $\begin{bmatrix} 5 & 18 \\ 39 & 68 \end{bmatrix}$

E. $\begin{bmatrix} 15 & 22 \\ 33 & 48 \end{bmatrix}$

40. Suppose that f is a function whose domain is all real numbers. Assume we know that $f(0) = -1$, $f(1) = 0$, and $f(2) = 1$. Which of the following statements might possibly be true?

I. f is a linear function.

II. f is a quadratic function.

III. f is a cubic function.

A. I only

B. III only

C. I and II

D. I and III

E. I, II, and III