FORTY-FOURTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Geometry 2000

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Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

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Forty-fourth Annual Mathematics Contest
Tennessee Mathematics Teachers Association
GEOMETRY 2000

Notation:

$\overline{AB}$ denotes the straight line segment joining points A and B.
$\triangle ABC$ denotes the triangle ABC.
$\angle AOB$ denotes the angle with vertex O determined by the points A, O and B, and a right angle (90° angle) is denoted by $\llcorner$.

Caution:

Many problems on this test are accompanied by diagrams. Though none are meant to be intentionally misleading, these diagrams may not be exactly to scale.

1. The area of an equilateral triangle is $2\sqrt{3}$. What is the length of one of its sides?

a) 1 b) 2 c) $2\sqrt{2}$ d) $2\sqrt{3}$ e) 4

2. Three circles are centered at different points A, B, and C. What is the greatest number of points that can be in common to two or more of the circles?

a) 3 b) 4 c) 6 d) 8

e) There can be infinitely many points in common between two of the circles.

3. Two of the angles in a triangle have measures 45° and 60°. What is the measure of the third angle?

a) 30° b) 55° c) 75° d) 90°

e) Cannot be determined without knowing the lengths of the sides.

4. In the figure, $\angle CAB$ is a right angle, D lies in the line segment $\overline{AB}$, the length of $\overline{AC}$ is 6, the length of $\overline{BC}$ is 10, and the length of $\overline{BD}$ is 4. What is the length of $\overline{CD}$?

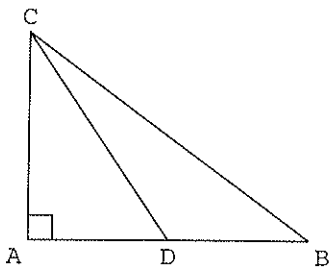
a) $4\sqrt{3}$

b) $2\sqrt{13}$

c) $2\sqrt{14}$

d) 8

e) $6\sqrt{2}$



5. Triangle $\triangle ABC$ is an acute triangle with area 126. The length of $\overline{AB}$ is 21 and the length of $\overline{AC}$ is 13. What is the length of $\overline{BC}$?

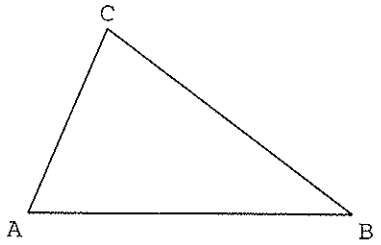
a) 16

b) 20

c) 21

d) 252/13

e) Not determined from the given information.



6. In the figure, $\overline{BC}$ is parallel to $\overline{DE}$. The length of $\overline{AB}$ is 6, the length of $\overline{BD}$ is 4, and the length of $\overline{AC}$ is 9. What is the length of $\overline{CE}$?

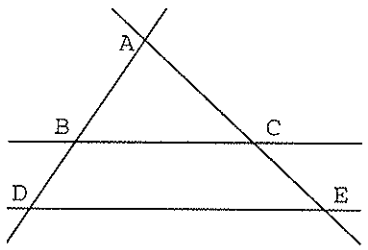
a) 9/2

b) 5

c) 6

d) 27/2

e) Not determined from the given information.



7. Triangles $\triangle ABC$ and $\triangle A'B'C'$ are such that $\angle ABC$ has twice the measure of $\angle A'B'C'$, $\angle BCA$ has twice the measure of $\angle B'C'A'$, and $\angle C'A'B'$ has twice the measure of $\angle CAB$. What is the measure of $\angle CAB$?

a) $22.5°$ b) $30°$ c) $45°$ d) $60°$

e) Not determined because the stated conditions are impossible.

8. Circles of radius 5 inches and 3 inches are centered 17 inches apart. What is the length of the line segment $\overline{AB}$ that crosses between the circles and is tangent to the two circles at points A and B?

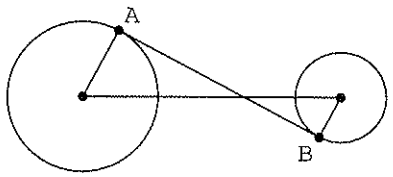
a) 15 inches

b) $\sqrt{353}$ inches

c) $\sqrt{323}$ inches

d) $\dfrac{\sqrt{325}+\sqrt{389}}{2}$ inches

e) $\sqrt{293}$ inches



9. Distinct points A, B, C, and D lie on a given circle in this order around the circle. The chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are all congruent. How many of the angles $\angle BAC$, $\angle CAD$, $\angle CBD$, $\angle BCA$, $\angle BDA$, and $\angle CDB$ must be congruent?

a) None are necessarily congruent.

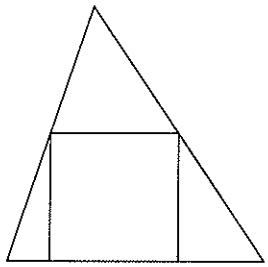
b) Exactly two must be congruent.

c) Exactly three must be congruent.

d) Exactly four must be congruent.

e) All six must be congruent.

10. A square is inscribed in an acute triangle, one side of the square lying along one side of the triangle. The altitude of the triangle to the side shared with the square is twice the length of one side of the square. What fraction of the area of the triangle is the area of the square?



a) $1/4$

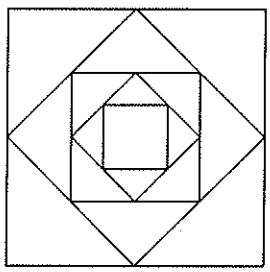
b) $1/\sqrt{3}$

c) $1/\sqrt{2}$

d) $1/2$

e) Cannot be determined from the given information.

11. In the figure, the vertices of each successive square are the midpoints of the sides of the next larger square. What fraction of the area of the largest square is the area of the smaller square?



a) $\frac{1}{2\sqrt{2}}$

b) $\frac{1}{4}$

c) $\frac{1}{4\sqrt{2}}$

d) $\frac{1}{8}$

e) $\frac{1}{16}$

12. In a particular rhombus, the four sides and one of the diagonals all have length 2. What is the length of the other diagonal?

a) 2 b) $2\sqrt{2}$ c) $2\sqrt{3}$ d) 4 e) $2\sqrt{5}$

13. Three sides of a regular hexagon, no two of which share a vertex of the hexagon, are extended to form a triangle. The perimeter of the triangle thus formed is how many times the perimeter of the original hexagon?

a) 3/2 b) 2 c) 5/2 d) 3 e) $2\sqrt{3}$

14. In a convex pentagon with vertices A, B, C, D, and E, the interior angle at A is 150°, the interior angle at B is 135°, the interior angle at C is 120°, and the interior angle at D is 90°. What is the interior angle at E?

a) 45° b) 65° c) 85° d) 105°

e) Cannot be determined because the stated conditions are impossible.

15. In a triangle with vertices A, B, and C, the length of $\overline{AB}$ is 5 and the length of $\overline{BC}$ is 3. Which one of the following must be true?

a) If the length of $\overline{AC}$ is 4, then $\angle ABC$ is a right angle.

b) Triangle $\triangle ABC$ is an acute triangle.

c) The length of $\overline{AC}$ is less than 8 and more than 2.

d) Triangle $\triangle ABC$ is not isosceles.

e) Triangle $\triangle ABC$ cannot be inscribed in a circle.

16. A quadrilateral $ABCD$ is inscribed in a circle. Which one of the following must be true?

a) $ABCD$ is a square.

b) $ABCD$ is a rectangle.

c) $ABCD$ is a rhombus.

d) $ABCD$ is a parallelogram.

e) Some angle of $ABCD$ is not obtuse.

17. Suppose distinct points A, B, C, and D are collinear with B and C both between A and D. Which one of the following is **not** necessarily true?

a) If $\overline{AB} \cong \overline{CD}$, then $\overline{AC} \cong \overline{BD}$.

b) If $\overline{AB} \cong \overline{BC} \cong \overline{CD}$ then B is between A and C.

c) Either B is between A and C, or B is between C and D.

d) If B is between C and D, then C is not between A and B.

e) If the midpoint of $\overline{AD}$ is also the midpoint of $\overline{BC}$ then $\overline{AB} \cong \overline{CD}$.

18. Consider a triangle $\triangle ABC$ and take D to be the midpoint of side $\overline{BC}$. Which one of the following statements is **not** equivalent to the condition that $\angle ABD \cong \angle ACD$?

a) The median $\overline{AD}$ is perpendicular to $\overline{BC}$.

b) The angle bisector of $\angle BAC$ passes through D.

c) $\overline{AB} \cong \overline{AC}$.

d) $\overline{BD} \cong \overline{DC}$.

e) $\triangle ABC \cong \triangle ACB$.

19. The line $\overline{AB}$ is perpendicular to the line $\overline{BC}$. A point D lies on $\overline{AC}$ such that the line $\overline{BD}$ is perpendicular to the line $\overline{AC}$. Which one of the following must hold?

a) $\angle DCB \cong \angle DBA$.

b) $\angle DBC$ and $\angle BAD$ are complementary.

c) $\angle DAB \cong \angle BCD$.

d) $\angle ABD$ and $\angle DBC$ are supplementary.

e) $\overline{BD}$ bisects $\angle ABC$.

20. Which one of the following four statements is **not** necessarily true in an arbitrary triangle, or is it the case that all are true?

a) The three medians of a triangle meet at a common point.

b) The three angle bisectors of a triangle meet at a common point.

c) The three altitudes of a triangle meet at a common point.

d) The three perpendicular bisectors of the sides of a triangle meet at a common point.

e) All of the above are true in any triangle.

21. In the plane, four points A, B, C and D are given with the line segments $\overline{AB}$ and $\overline{DC}$ parallel and of the same length but not on the same line. Assuming that the line segments $\overline{AC}$ and $\overline{BD}$ intersect at a point F interior to both line segments, which one of the following does **not** necessarily follow?

a) The point F is the midpoint of both $\overline{AC}$ and $\overline{BD}$.

b) $\angle ABD \cong \angle CDB$

c) The line segments $\overline{BC}$ and $\overline{DA}$ are parallel and the same length.

d) The line segments $\overline{AC}$ and $\overline{BD}$ are perpendicular.

e) Angles $\angle DAB$ and $\angle ABC$ are supplementary.

22. In a triangle $\triangle ABC$, the length of $\overline{AB}$ is 8, the length of $\overline{AC}$ is 5 and the measure of $\angle ABC$ is 30°. What is the length of $\overline{BC}$?

a) 7 b) $4\sqrt{3}-3$ c) $4\sqrt{3}$ d) $4\sqrt{3}+3$

e) Not determined from the given information.

23. A trapezoid has parallel bases with lengths 4 and 6 and the other two sides are each of length 2. What is the area of the trapezoid?

a) 5 b) $5\sqrt{2}$ c) $5\sqrt{3}$ d) 10

e) 0, because the trapezoid is degenerate and lies along a line.

24. The three altitudes of an equilateral triangle $\triangle ABC$ meet at a point G. If the length of $\overline{AB}$ is 6, what is the length of $\overline{AG}$?

a) $\sqrt{3}$ b) 3 c) $2\sqrt{3}$ d) 4 e) $3\sqrt{3}$

25. Perpendicular line segments $\overline{AB}$ and $\overline{CD}$ intersect in a point E. The length of $\overline{AB}$ is 8 and the length of $\overline{CD}$ is 6. What is the area of quadrilateral $ACBD$?

a) 6 b) 12 c) 24 d) 48

e) Not determined from the given information.

26. In the figure, A, B, C, and D lie on a circle and $\overline{AC}$ intersects $\overline{BD}$ at E. The length of $\overline{AE}$ and the length of $\overline{EC}$ are both 6. The length of $\overline{BD}$ is 13. The length of $\overline{BE}$ is greater than the length of $\overline{ED}$. What is the length of $\overline{BE}$?

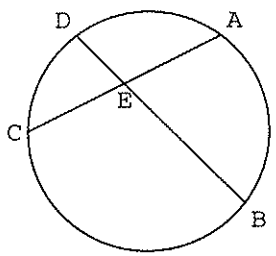
a) 4

b) 7

c) 8

d) 9

e) 10



27. In the acute triangle $\triangle ABC$, the line segments $\overline{AD}$ and $\overline{BE}$ are altitudes. If the length of $\overline{AB}$ is 10, the length of $\overline{CD}$ is 2, and the length of $\overline{AD}$ is 6, what is the length of $\overline{BE}$?

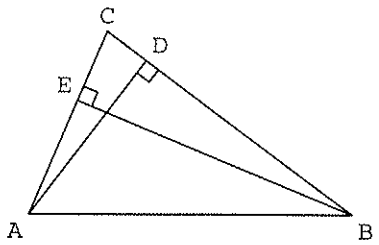
a) 8

b) 12

c) $2\sqrt{10}$

d) $3\sqrt{10}$

e) Not determined from the given information.



28. Circles are centered at collinear points A, B, and C with B between A and C. The circles centered at A and C are externally tangent to the circle centered at B. The three circles have a common tangent line. If the radius of the circle centered at A is 9, and the radius of the circle centered at B is 6, what is the radius of the circle centered at C?

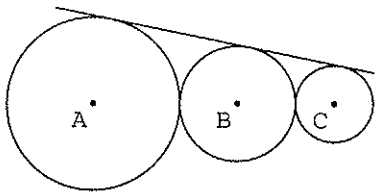
a) 3

b) 4

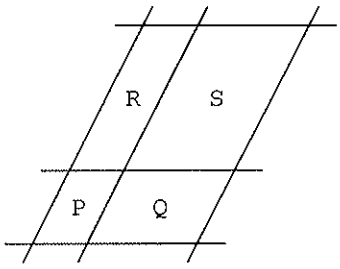
c) $2\sqrt{3}$

d) $3\sqrt{2}$

e) $3+\sqrt{6}$



29. In the figure, three parallel lines interesect three other parallel lines to form four bounded regions P, Q, R, and S as labelled. If regions Q and R each have twice the area of region P, what is the area of region S?



a) 3 times the area of region P.

b) 5/9 of the total area of all four regions.

c) Equal to the area of region Q.

d) The total area of regions R and Q.

e) Not determined from the given information.

30. Which one of the following is **not** sufficient to determine the area of a triangle?

a) The lengths of the three sides of the triangle.

b) The length of any side and any altitude of the triangle.

c) The radius of the inscribed circle and the perimeter of the triangle.

d) The lengths of the three altitudes of the triangle.

e) The lengths of two sides and the measure of the angle between these sides.

31. In a city with square blocks set out in a rectangular grid with north-south and east-west streets (none one-way), the main post office is 4 blocks east and 2 blocks north of city hall, the library is 10 blocks due south of the main post office, and the arena is 4 blocks east and 2 blocks north of the library. Then from city hall to the arena is a distance of

a) 10 blocks in a straight line but 14 blocks following streets.

b) $8\sqrt{2}$ blocks in a straight line but 16 blocks following streets.

c) $2\sqrt{41}$ blocks in a straight line but 18 blocks following streets.

d) $4\sqrt{5}+10$ blocks in a straight line but 22 blocks following streets.

e) $2\sqrt{65}$ blocks in a straight line but 24 blocks following streets.

32. Suppose two pyramids are similar and the surface area of the larger is 16 times the surface area of the smaller. How many times the volume of the smaller pyramid is the volume of the larger pyramid?

a) 4 b) 8 c) 16 d) 32 e) 64

33. What is the length, in feet, of the diagonal of a rectangular box having height 3 feet, width 4 feet, and length 5 feet?

a) 5 b) $5\sqrt{2}$ c) $2\sqrt{15}$ d) $\sqrt{73}$ e) 10

34. Let A be the vertex of a right circular cone with base centered at O and let B be a point on the circumference of its circular base. If the length of $\overline{AO}$ is 10 and length of $\overline{AB}$ is 14, what is the lateral surface area of the cone?

a) 70π b) $80\pi\sqrt{3}$ c) $100\pi\sqrt{2}$ d) 140π e) $56\pi\sqrt{6}$

35. A propane tank is in the shape of a cylinder with hemispherical end caps. The cylinder has diameter 4 feet and the total length of the tank is 12 feet. What is the volume of the tank in cubic feet?

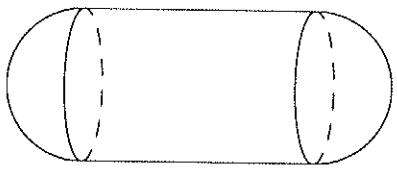
a) $\frac{32\pi}{3}$

b) $\frac{128\pi}{3}$

c) $\frac{176\pi}{3}$

d) 48π

e) $\frac{1024\pi}{3}$



36. Three cylinders, each of diameter 6 inches, are tightly bound together with a string. What is the length of the string (not counting the knot or the free ends) in inches?

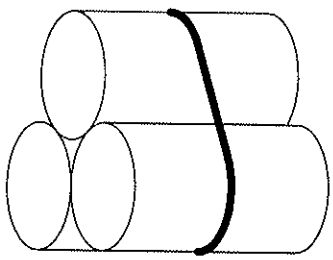
a) $6 + 6\pi$

b) $6 + 12\pi$

c) $18 + 6\pi$

d) $18 + 12\pi$

e) $36 + 12\pi$



37. A square $ABCD$ having area 4 is rotated 45° about its center to a new position $A'B'C'D'$. What is the area of the octagon formed by intersecting the interiors of $ABCD$ and $A'B'C'D'$?

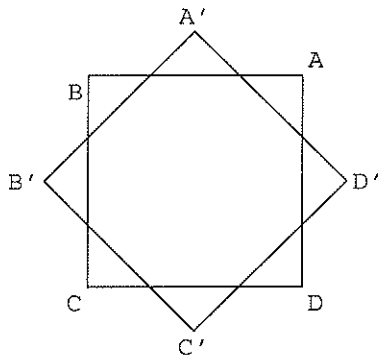
a) 2

b) $4\sqrt{2}-3$

c) $2\sqrt{2}$

d) 3

e) $8\sqrt{2}-8$



38. A field is 120 feet wide by 160 feet long. Runners will start a race at a point P near one corner of the field 20 feet from each side in that corner. The runners want to race around the field along a path of minimal length that reaches each of the four sides of the field and returns to the starting point (the illustrated path is not minimal length). Which of the following is closest to the length of a shortest such path?

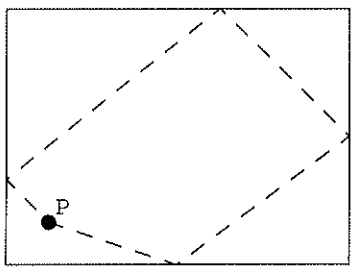
a) 280 feet

b) 400 feet

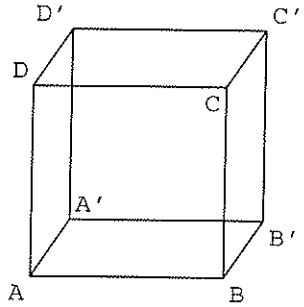
c) 450 feet

d) 500 feet

e) 560 feet



39. A cube has parallel square faces $ABCD$ and $A'B'C'D'$ that are translates as illustrated. What is the dihedral angle ($\leq 90°$) between the plane containing $ABC'D'$ and the plane containing $AA'C'C$?



a) 30°

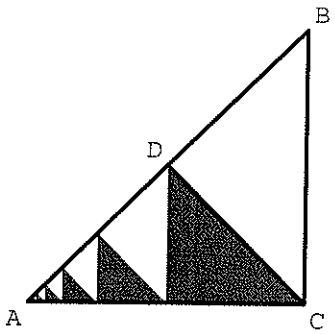
b) 45°

c) 60°

d) 90°

e) None of the above.

40. In the figure, an isosceles right triangle $\triangle ABC$ is subdivided by dropping an altitude from the right angle at C to a point D on the hypotenuse. The resulting right triangle ACD is similarly subdivided by droping a perpendicular to its hypotenuse and this process is continued indefinitely always taking A as a vertex of the triangle next to be subdivided. The triangles which have a side on the line segment $\overline{AC}$ are shaded. If the length of $\overline{AC}$ is 1, what is the total area of all the shaded triangles?



a) 1/8

b) 1/6

c) 1/4

d) 1/3

e) 1/2