TWELFTH ANNUAL MATHEMATICS CONTEST

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COMPREHENSIVE TEST

1968

Scoring Formula: 4R - W.

Prepared by:

James R. Wesson, Chairman
Roger Cooke
Matthew Gould
Ann Noonburg
Robbie Shackleford

- Vanderbilt University

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work the problems, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. A sample follows:

1. If $2x = 3$, then x equals:

   (1) $\frac{2}{3}$ (2) 3 (3) 6

   (4) $\frac{3}{2}$ (5) none of these.

1. 

The correct answer for sample (1) is "$\frac{3}{2}$", which is answer (4); so you would answer this problem by making a heavy black mark under space 4 as indicated above.

If you should change your mind about an answer, be sure to erase completely. Avoid wild guessing, as wrong answers count against you. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. Three corners of a rectangle are the points (1,4), (7,4), and (1,8). The diagonals of the rectangle meet at the point

   (1) (4,2)

   (2) (4,6)

   (3) (3,1)

   (4) (5,6)

   (5) none of these.

2. To which of the following is $\frac{a^{-2}y^{-2}}{a^{-2}+y^{-2}}$ equal?

   (1) a^2y^{-2}

   (2) 1

   (3) $\frac{1}{y^2 + a^2}$

   (4) $y^{-2} + a^{-2}$

   (5) $\frac{y}{a^2 + y^2}$.

3. The slope of the line through the two points (-3,4) and (1,-2) equals

   (1) $\frac{-3}{2}$ (2) $\frac{2}{3}$ (3) -1 (4) $\frac{-2}{3}$ (5) $\frac{3}{2}$.

4. If i represents $\sqrt{-1}$, then i^{291} equals

   (1) i (2) -i (3) -1 (4) $-i^3$ (5) 1 + i .

5. If the hypotenuse of a right triangle is twice as long as one leg of the triangle, then the angle opposite that leg equals

   (1) 45°

   (2) $\frac{\pi}{3}$

   (3) $22\frac{1}{2}^\circ$

   (4) 30°

   (5) none of these.

6. If a is rational and b irrational, then a + b is

   (1) sometimes rational and sometimes irrational.

   (2) always rational.

   (3) always irrational.

   (4) an improper fraction.

   (5) the square root of a negative number.

7. If a sphere is doubled in volume, then its diameter is multiplied by

   (1) 2

   (2) 8

   (3) $\sqrt[3]{2}$

   (4) $\frac{4}{3}\pi\sqrt{2}$

   (5) none of these.

8. Let a, b be real. The absolute value of b -a (written $|b - a|$) is

   (1) always positive.

   (2) always equal to $|a - b|$.

   (3) always zero.

   (4) always equal to a - b .

   (5) none of these.

9. The formula $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ is true

   (1) for all values of a and b.

   (2) if and only if a and b are positive.

   (3) for no values of a and b .

   (4) only if either a = 0 or b = 0.

   (5) if a,b and a + b are perfect squares.

10. If K is a positive constant, which of the following equations states that x varies directly as y and inversely as z? (Assume x, y, z positive.)

    (1) $xz = Ky$

    (2) $Kx = yz$

    (3) $x = Kyz$

    (4) $xyz = K$

    (5) $xy = Kz$.

11. The sum of two rational numbers is

    (1) sometimes rational and sometimes irrational.

    (2) always rational.

    (3) always irrational.

    (4) an improper fraction.

    (5) a proper fraction smaller than 2.

12. If $A = \{a, b, c, d, e\}$ and $B = \{a, e, i, o, u\}$, then $A \cap B =$

    (1) $\{a, b, c, d, e, i, o, u\}$

    (2) $\emptyset$, the empty set.

    (3) $\{a, e\}$

    (4) $\{a\}$

    (5) none of these.

13. Which one of the following is positive?

    (1) $\cos 90^\circ$

    (2) $\cot 120^\circ$

    (3) $\tan 0^\circ$

    (4) $\sin 185^\circ$

    (5) $\cos (-40^\circ)$.

14. If $\sin A \neq 0$, then $\frac{\sin 2A}{\sin A}$ equals

(1) 2

(2) sin A

(3) 2 cos A

(4) 2 sin A

(5) none of these.

15. The solution set of $\frac{1}{x - 2} > 0$ is the set of all real numbers x such that

(1) $x > 2$

(2) $0 < x < 2$

(3) $x < 2$

(4) $x < -2$

(5) none of these.

16. $\sin 60^\circ + \sin 30^\circ =$

(1) $\sin 90^\circ$

(2) $\frac{1}{\sqrt{3} - 1}$

(3) $\frac{2}{3}$

(4) $\frac{\sqrt{3} - 1}{2}$

(5) none of these.

17. If a is neither 0 nor 1 nor -1, then the quotient $\left(\frac{a - a^2}{a^2 - 1}\right) \div \left(\frac{a}{a + 1} - a\right)$ is equal to

(1) $\frac{a}{1 + a}$

(2) $\frac{-1}{a}$

(3) $\frac{1}{1 - a}$

(4) $\frac{1}{a}$

(5) $\frac{1}{a - 1}$.

18. The roots of $4x^2 - 3x = 5$ are

(1) imaginary.

(2) real, equal, and rational.

(3) real, unequal, and irrational.

(4) real, unequal, and rational.

(5) none of these.

19. Let $\tan \alpha = m_1$ and $\tan \beta = m_2$. If $\tan (\alpha + \beta)$ exists, it is equal to

(1) $\dfrac{m_1 - m_2}{m_1 + m_2}$

(2) $\dfrac{m_1 m_2}{1 + m_1}$

(3) $m_1 + m_2$

(4) $\dfrac{m_1 - m_2}{1 + m_1 m_2}$

(5) $\dfrac{m_1 + m_2}{1 - m_1 m_2}$

20. An equilateral triangle with side s is drawn on the base of a square with side s. The perpendicular distance x from the vertex to the side of the square is:

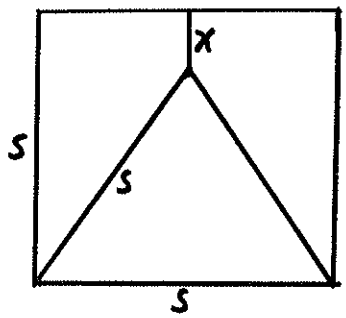
(1) $1 - \frac{1}{2} s \sqrt{3}$

(2) $\left(\dfrac{2 - \sqrt{3}}{2}\right) s$

(3) $\frac{1}{2} s$

(4) $\dfrac{\sqrt{3}}{2} s$

(5) none of these.



21. A number 11111 to the base 2 is equal to what number to the base 10?

(1) 62 (2) 32 (3) 22 (4) 31 (5) none of these.

22. What is the third term of an arithmetic progression consisting of 26 terms, given that the first term is 2 and the last term is 402?

(1) 18 (2) 34 (3) 32 (4) 50 (5) 48.

23. An equilateral triangle of area $5\sqrt{3}$ has a side of length

(1) $2\sqrt{5}$ (2) $3\sqrt{3}$ (3) $2\sqrt{3}$ (4) $\sqrt{5}$ (5) $\frac{1}{2}\sqrt{5}$.

24. The parabola $y = x^2 - 12x + 39$ has its vertex at the point

(1) (12,39) (2) (0,39) (3) (3,6) (4) (-6,147) (5) (6,3).

25. 2 sin A cos B is always equal to

(1) $\sin(A + B) + \sin(A - B)$.

(2) $\cos(A + B) + \cos(A - B)$.

(3) $\sin(A + B) - \sin(A - B)$.

(4) sin 2AB.

(5) the average of sin 2A and sin 2B.

26. The equations $x + y = \sqrt{2}$, $x^2 + y^2 = 1$ have

(1) no common solutions.

(2) one common solution.

(3) two common solutions.

(4) four common solutions.

(5) an infinite number of solutions.

27. Let i be the usual imaginary square root of -1. A complex number whose square is equal to -i is

(1) -1 (2) i^3 (3) -i (4) $(\cos 45^\circ) - (\sin 45^\circ)i$. (5) none of these.

28. A container in the shape of a right circular cone is used to fill another container in the shape of a cylinder. Both containers have the same radius. If the cone has a height of 5 inches, and the cylinder has a height of 10 inches, how many times must the cone be filled and emptied into the cylinder in order to fill the cylinder?

(1) 3 (2) 2 (3) 6 (4) The answer depends on the radius.

(5) None of these.

29. For the equation $\frac{x}{x^2 - 1} - \frac{1}{x^2 - 1} + \frac{2}{x + 1} = 0$

(1) $x = 1$ is a solution.

(2) $x = -1$ is a solution.

(3) $x = 0$ is a solution.

(4) none of the above is a solution, but there is a real solution.

(5) the equation has no real solution.

30. The conjugate of the complex number $a + bi$ is defined as $a - bi$. (Here a, b are real, and $i^2 = -1$.) Consider the following two statements: FIRST: The conjugate of the sum of two complex numbers must equal the sum of their conjugates. SECOND: The conjugate of the product of two complex numbers must equal the product of their conjugates.

(1) The two statements are contradictory.

(2) The first statement is true, the second false.

(3) The first statement is false, the second true.

(4) Both statements are false.

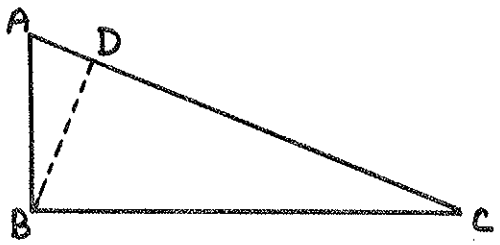
(5) Both statements are true.

31. The right triangle ABC has area equal to 10. If leg BC has length 5, find the length of the altitude BD.

(1) $\frac{20}{\sqrt{29}}$

(2) $\frac{7}{2}$

(3) $\frac{20}{\sqrt{41}}$



(4) 3

(5) none of these.

32. If x and y are real numbers, which of the following statements is true?

(1) $\frac{x}{x + y} = 1 + \frac{x}{y}$

(2) If $\sin(x + y) > 0$, then $\sin x + \sin y > 0$.

(3) $x^y + y^x = (xy)^{xy}$

(4) If $x^2 - 2x + 2 = 0$, then $x^4 + 4 = 0$.

(5) none of these.

33. The two roots of the quadratic equation $ax^2 + 6x + 3 = 0$ have a sum of 1. The product of the two roots is

(1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) 3 (4) -3 (5) none of these.

34. If the sine of angle A is equal to $|x|$ and $\sin 2A < 0$, then $\tan(\frac{A}{2})$ is

(1) $\dfrac{x}{2(1 - x^2)}$

(2) $\dfrac{|x|}{2\sqrt{1 - x^2}}$

(3) $\dfrac{1 - \sqrt{1 - x^2}}{|x|}$

(4) $\dfrac{1 + \sqrt{x^2 - 1}}{x}$

(5) $\dfrac{1 + \sqrt{1 - x^2}}{|x|}$

35. Consider the following three statements concerning sets A, B, C.

I. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
II. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
III. $(A \cap B) \cup C = A \cap (B \cup C)$

Then:

(1) Only statement I is true.

(2) Only statement II is always true.

(3) Only statement III is always true.

(4) More than one of the statements are true.

(5) All three statements are true.

36. If $\sin A = \frac{12}{13}$ and $\csc 2A < 0$, then $\tan 2A$ is equal to

(1) $\frac{120}{119}$ (2) $\frac{-120}{119}$ (3) $\frac{12}{5}$ (4) $\frac{-12}{5}$ (5) none of these.

37. For x,y positive integers, define x*y to be the greatest common divisor of x and y. Then

(1) The commutative law x*y = y*x always holds, but the associative law (x*y)*z = x*(y*z) may fail.

(2) x*1 = x for all x.

(3) If x*y = x*z, then y = z.

(4) None of the above is true.

(5) More than one of (1), (2), (3) are true.

38. The number $9^n + 9^n + 9^n$ can be written as

(1) $24(4n^2 - 3n) + 3$.

(2) $(27)^{\frac{n}{3}}$

(3) 9^{3n}

(4) 27^{3n}

(5) $(27)^{\frac{2n+1}{3}}$

39. The determinant $\begin{vmatrix} 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & a & a^2 \end{vmatrix}$ is equal to 30.

Then the only value(s) possible for a is (are)

(1) 10 and -1

(2) -1

(3) 10 and 1

(4) -10 and 1

(5) not given above.

40. If 4 marbles are drawn simultaneously from a bag containing 5 red marbles and 6 white marbles, what is the probability that 2 of the marbles selected are red and the other two are white?

(1) $\frac{5}{11}$ (2) $\frac{151}{330}$ (3) $\frac{1}{2}$ (4) $\frac{4}{11}$ (5) $\frac{9}{32}$.