TENTH ANNUAL MATHEMATICS CONTEST

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THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

COMPREHENSIVE TEST

1966

Scoring Formula: 4R - W.

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DIRECTIONS:

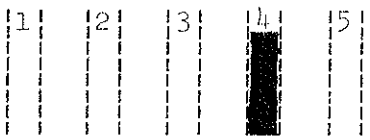
Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work the problems, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. A sample follows:

1. If $2x = 3$, then x equals:

(1) 2/3; (2) 3; (3) 6;

(4) 3/2; (5) none of these.



The correct answer for the sample problem is "3/2", which is answer (4); so you would answer this problem by making a heavy black mark under space 4 as indicated above.

If you should change your mind about an answer, be sure to erase completely. Avoid wild guessing, as wrong answers count against you. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. The sum of the odd numbers between 1 and 100 is:
   (1) 5050
   (2) 2500
   (3) 4986
   (4) 11,942
   (5) 2000.

2. If $f(x) = x^3 - 3x^2 - 30x + 4$ then $f(13)$ is equal to:
   (1) 1304
   (2) 16936
   (3) - 178
   (4) 0
   (5) None of the above.

3. If $\log_b 27 = -3$, then b equals:
   (1) 3
   (2) 10
   (3) .1
   (4) 1/3
   (5) π.

4. A parallelogram ABCD has AB = 3 and BC = 4. If $\angle$ DAB = 60°, then the area of the parallelogram is:
   (1) 6
   (2) $6\sqrt{3}$
   (3) 4
   (4) $3\sqrt{3}/2$
   (5) Not enough information to determine the area.

5. For what values of k does the equation $9x^2 - 2kx + (k - 2) = 0$ have equal roots?
   (1) Only for k = 6
   (2) Both for k = 4 and k = -4
   (3) Both for k = 6 and k = 3
   (4) For no value of k
   (5) For $k = \sqrt{-7}$ only.

6. Suppose A and B are sets. Then $A \cap (B \cup A)$ is always the same set as:
   (1) A
   (2) B
   (3) $A \cup B$
   (4) $A \cap B$
   (5) None of the above.

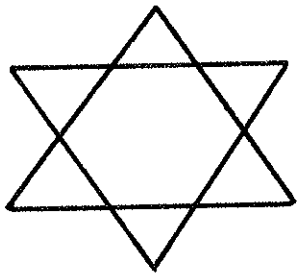
7. The coefficient of x^4y^2 in the expansion of $(x + y)^6$ is:
   (1) 15
   (2) 6
   (3) 112
   (4) 12
   (5) 36
   (6) 0.

8. How many members does the solution set of
$$x + 2 = \sqrt{4 + x\sqrt{8 - x}}$$
contain?
   (1) 0
   (2) 1
   (3) 2
   (4) 3
   (5) The solution set is infinite.

9. The solution set of the system
$$4x + 6y = 3$$
$$2x + 3y = 7$$
when plotted in a co-ordinate plane will be:
   (1) Empty
   (2) A single point
   (3) A line
   (4) The whole plane
   (5) None of the above.

10. What must be true about a real number r in order that $|-r| = -r$?
   (1) $r < 1$
   (2) $r > 0$
   (3) $r = 0$
   (4) $r \leq 0$
   (5) $r \geq 0$.

11. A star of David with six points is formed by extending the sides of a regular hexagon as shown below:



If the hexagon has side a, then the measure of the area of the star is:

(1) $6a^2$

(2) $3\sqrt{3}\ a^2$

(3) $\frac{\sqrt{3}}{4}\ a^2$

(4) πa^2

(5) None of the above.

12. An operation $*$ is defined on the set of rational numbers as follows:

$$a * b = a + b - ab.$$

For a fixed rational number a, consider the equation $a * x = 0$. Which of the following is true?

(1) The equation is always solvable, regardless of the value of a.

(2) Except for one special value for a, the equation is never solvable.

(3) Except for one special value for a, the equation is always solvable.

(4) The equation is never solvable, regardless of the value of a.

(5) None of the above statements is true.

13. The set of numbers x for which $|x + 4| < 2$ is:

(1) $\{x : -6 < x < -2\}$

(2) $\{x : 2 < x < 6\}$

(3) $\{x : 0 < x < 8\}$

(4) $\{x : -6 \leq x \leq -2\}$

(5) $\{x : 2 \leq x \leq 6\}$.

14. The numeral in the binary system for a certain integer is 101011001. In the base eight system, the numeral for this number is:

(1) 327

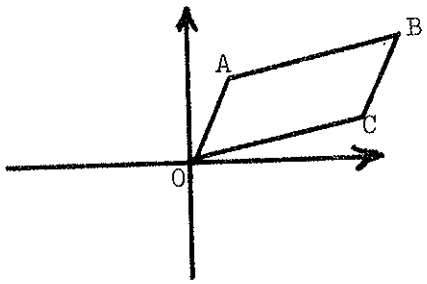
(2) 521

(3) 531

(4) 3211

(5) 345.

15. A co-ordinate system is set up in a plane. A parallelogram OABC is situated as shown in the plane with one vertex at the origin



If A has co-ordinates (x_1,y_1) and C has co-ordinates (x_2,y_2), then the measure of the area of $\square$ OABC is given by:

(1) $x_1y_2 - x_2y_1$

(2) $x_2y_1 - x_1y_2$

(3) $x_1y_1 + x_2y_2$

(4) $x_1^2 + x_2^2 + y_1^2 + y_2^2$

(5) $x_1x_2 + y_1y_2$.

16. The set of numbers x for which $x^2 - x - 2 < 0$ is:

(1) $\{x : x < -1 \text{ or } x > 2\}$

(2) $\{x : x > -1 \text{ and } x < 2\}$

(3) $\{x : x > -1 \text{ or } x < 2\}$

(4) $\{x : x \leq 1 \text{ or } x \geq 2\}$

(5) $\{x : x \geq -1 \text{ and } x \leq 2\}$.

17. If $x + y = 3$ and $x^2 + y^2 = 5$, then xy is equal to:

(1) 15

(2) -4

(3) 2

(4) 0

(5) None of the above.

18. A right circular cone has altitude 12 ft. and radius of base 6 ft. A cone is inscribed with its vertex at the center of the base of the given cone and its base parallel to the base of the given cone. Find the volume of the inscribed cone if its altitude is 4 ft.

(1) $\frac{64\pi}{3}$ (2) 32π (3) $\frac{16\pi}{3}$ (4) $\frac{48\pi}{3}$ (5) None of these.

19. A card is drawn from a standard deck of 52 playing cards. A second card is drawn without replacing the first card. What is the probability of successively drawing two aces?

(1) 1/221
(2) 3/676
(3) 3/12,946
(4) 1/5
(5) None of the above.

20. The quotient $\frac{3 - 5i}{5 + 3i}$ where $i = \sqrt{-1}$ is equal to:

(1) $\frac{3}{5} - \frac{5}{3} i$
(2) $\frac{3 - 5i}{34}$
(3) $-i$
(4) i
(5) $7 + 16i$.

21. If x is close to but not equal to 2, then

$$\frac{x^2 + 2x - 8}{x - 2}$$

is closest to:

(1) 0
(2) 2
(3) -4
(4) 6
(5) 4.

22. Suppose that $|x - 2| < 1$. Then $|x^2 + x - 6|$ will be less than a positive number e whenever $|x - 2|$ is less than

(1) $2e$
(2) $e/4$
(3) e
(4) $1/6$
(5) $1/4$.

23. If four of the letters of TENNESSEE are selected one at a time and without replacements, what is the probability that in the order they are drawn they spell SENT?

(1) $\frac{1}{27}$

(2) $\frac{4}{2187}$

(3) $\frac{1}{189}$

(4) $\frac{1}{378}$

(5) None of these.

24. A co-ordinate system is set up in a plane. The solution set of the system of inequalities

$$x^2 + y^2 \leq 1$$
$$x + y \geq 1$$

when plotted in the plane determines a geometric figure. The area of this figure (in terms of the square units determined by the units of the co-ordinate system) is:

(1) $\frac{1}{3}$

(2) $\frac{3\pi}{4}$

(3) $4 - \sqrt{2}$

(4) 6π

(5) $\frac{1}{4}(\pi - 2)$.

25. Let f be the function defined by the formula $f(t) = t^2 - 2$. The slope of the line joining the points $(1, f(1))$ and $(-2, f(-2))$ is:

(1) -1

(2) 1

(3) 0

(4) -3

(5) 3.

26. $\dfrac{(x + h)^{-1/2} - x^{-1/2}}{h}$ is equal to

(1) $h^{-3/2}$

(2) $-[x(x + h)^{1/2} + x^{3/2} + hx^{1/2}]^{-1}$

(3) $[x(x + h)^{1/2} + x^{3/2} + hx^{1/2}]^{-1}$

(4) $[(x^2 + x)^{1/2} + x^{3/2} + hx^{1/2}]$

(5) None of these.

27. The sum of the first 50 even integers is

(1) 2652

(2) 2570

(3) 2550

(4) 2500

(5) 2682.

28. How many solutions are there to the following system of equations

$$\begin{aligned} 2x - y &= 5 \\ x + y + 3z &= 1 \\ x - y - z &= 3 \end{aligned}$$

(1) None

(2) 1

(3) 2

(4) infinitely many

(5) None of these.

29. A co-ordinate system is set up in a plane. The solution set of the set of inequalities

$$\begin{aligned} x &> 0 \\ y &> 0 \\ 2x-3y &> 0 \\ 2-\ 2x+3y &> 0 \end{aligned}$$

when plotted in the plane looks like:

(1) a triangle

(2) a quadrilateral

(3) an infinite strip

(4) a circle

(5) None of the above.

30. The set of points (x,y) satisfying $-2\pi \leq x \leq 2\pi$ at which the curves $y = x$ and $y = x + \sin x$ intesect is

(1) $\{(0,0)\}$

(2) $\{(0,0), (\frac{\pi}{2}, \frac{\pi}{2} + 1), (\pi,\pi), (2\pi, 2\pi)\}$

(3) $\{(0,0), (\pi,\pi), (2\pi, 2\pi)\}$

(4) $\{(-2\pi, -2\pi), (-\pi, -\pi), (0,0)\}$

(5) $\{(-2\pi, -2\pi), (-\pi, -\pi), (0,0), (\pi,\pi), (2\pi, 2\pi)\}$.

31. A rectangle has an area of 25 and perimeter of 25. What is the length of the longest side?

(1) 10

(2) 4

(3) 5/2

(4) 5/4

(5) 2.

32. If

$$\sin\theta = \frac{a^2 - b^2}{a^2 + b^2}$$

$$\cos\theta = \frac{2ab}{a^2 + b^2}$$

where $a \neq -b$, then $\tan(\theta/2)$ is given by:

(1) a/b

(2) $(a - b)/(a + b)$

(3) $1/2$

(4) $2ab/(a^2 - b^2)$

(5) None of the above.

33. The product of the roots of $4x^3 - 8x^2 - 16x + 5 = 0$ is

(1) -2

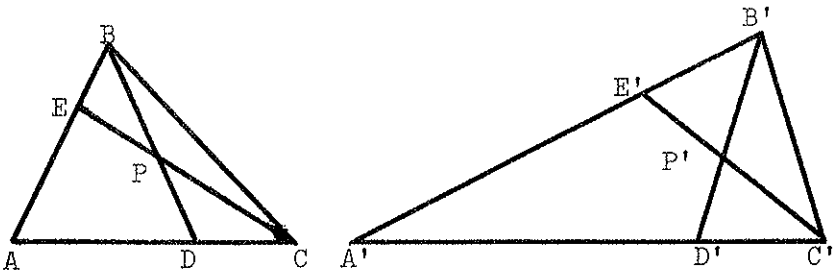
(2) $-5/4$

(3) 4

(4) -5

(5) 8.

34. Consider two figures:



Assume that AE:EB = A'E' : E'B' and that AD:DC = A'D' : D'C'. Which of the following is the most general assumption under which it will also be true that BP:PD = B'P' : P'D' and EP:PC = E'P' : P'C' ?

(1) Triangles ABC and A'B'C' are congruent

(2) Triangles ABC and A'B'C' are similar

(3) $\angle$ ABC = $\angle$ A'B'C'

(4) AB : AC = A'B' : A'C'

(5) No additional assumption is required.

35. A farmer wishes to fence a rectangular area out of a field one side of which is adjacent to a straight segment of roadway and is already fenced. If he has p feet of wire, what should be the length of the side adjacent to the roadway in order that he have the maximum area?

(1) p/3 ft.

(2) p/5 ft.

(3) $\frac{p^2}{8}$ ft.

(4) $\frac{3p}{8}$ ft.

(5) None of the above.

36. The cosine of $5\pi/12$ is

(1) $\frac{\sqrt{6}+\sqrt{2}}{4}$

(2) $\frac{\sqrt{6}-\sqrt{2}}{4}$

(3) $\frac{\sqrt{2}-\sqrt{6}}{2}$

(4) $\frac{\sqrt{6}+\sqrt{2}}{2}$

(5) None of the above.

37. Consider the following operations "*" defined on the set of integers: (A) $a * b = a$, (B) $a * b = b$, (C) $a * b = a - b$, (D) $a * b = b - a$, and (E) $a * b = a + b - ab$. Which of these operations are commutative?
(1) C and D
(2) A, B and E
(3) All of them
(4) C, D and E
(5) Only E.

38. Which of the operations defined in the preceding problem are associative?
(1) C and D
(2) A, B and E
(3) All of them
(4) C, D and E
(5) Only E.

39. For which of the operations defined in problem 37 is there an identity element?
(1) C and D
(2) A, B and E
(3) None of them
(4) C, D and E
(5) Only E.

40. Suppose $a = 212$ and $b = 2201$ are numbers expressed in the ternary (base three) system. The product ab expressed in the ternary system is:
(1) 11102212
(2) 10002211
(3) 22110001
(4) 2022012
(5) None of these.