EIGHTH ANNUAL MATHEMATICS CONTEST

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THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

COMPREHENSIVE TEST

1964

Scoring Formula: 4R - W

Prepared by:

Donald J. Dessart (Chairman)
James Cooley
Edgar Eaves
Herbert Lee

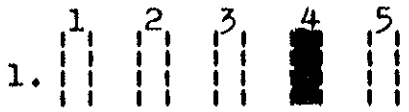
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work the problems, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. A sample follows:

1. If $2x = 3$, then x equals:

   (1) 2/3; (2) 3; (3) 6;
   (4) 3/2; (5) none of these.



The correct answer for the sample problem is "3/2", which is answer (4); so you would answer this problem by making a heavy black mark under space 4 as indicated above.

If you should change your mind about an answer, be sure to erase completely. Avoid wild guessing, as wrong answers count against you. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. If $a + b = 1$, then the largest value of ab is:

(1) 1 ; (2) 1/2 ; (3) $\sqrt{2}$; (4) 1/4 ;

(5) none of the above.

2. The set of real numbers is a subset of:

(1) the set of rational numbers.
(2) the set of irrational numbers.
(3) the set of complex numbers.
(4) the set of integers.
(5) none of the above.

3. The star Sirius is approximately 51,200,000,000,000 miles from the earth. In scientific notation, this distance would be written as:

(1) 5.12×10^{13} miles; (2) 5.12×10^{12} miles;

(3) 5.12×10^{11} miles; (4) 51.2×10^{12} miles;

(5) none of the above.

4. Which statement may be true concerning the roots of the equation:

$$2x^3 + 5x^2 - 3x + 4 = 0$$

(1) it has three imaginary roots.
(2) it has exactly four roots.
(3) it has two imaginary roots and one negative root.
(4) it has two negative roots and one positive root.
(5) it has two negative roots and one imaginary root.

5. The solution of the equations $\begin{cases} 2a - 3b = 7 \\ 4a - 6b = 20 \end{cases}$ is:

(1) $a = 18$, $b = 12$; (2) $a = 0$, $b = 0$; (3) unlimited number of solutions;

(4) no solution; (5) $a = 8$, $b = 5$

6. Suppose the discriminant of $ax^2 + 2bx + c = 0$ is zero, then we can make the following true statement about a, b, c.

(1) they form an arithmetic progression.
(2) they form a geometric progression.
(3) b is zero, if a and c are positive.
(4) b is not zero, if a and c are zero.
(5) none of the above is true.

7. What is the set of real numbers for x such that

$$x^2 - 3x > -2 .$$

(1) $\{x \mid x > 2 \text{ or } x < 1\}$; (2) $\{x \mid x > -2 \text{ or } x < -1\}$;

(3) $\{x \mid x > 2 \text{ and } x < 5\}$; (4) $\{x \mid x > 2 \text{ and } x < 1\}$;

(5) none of the above.

8. If $f(x) = \frac{x(x+1)}{2}$, then $f(x+1)$ equals:

(1) $\frac{x^2 - 3x + 2}{2}$; (2) $\frac{(x-1)(x-2)}{2}$; (3) $\frac{(x-1)(x+2)}{2}$

(4) $\frac{f(x)}{x}$; (5) $\frac{(x+2)f(x)}{x}$.

9. Solve $\log_{10}x + \log_{10}(2x - 1) = 2$ for x. The solution is:

(1) $\frac{1 \pm \sqrt{801}}{4}$; (2) $\frac{1 + \sqrt{801}}{4}$; (3) $\frac{1 + 3\sqrt{801}}{4}$;

(4) $\frac{\sqrt{801}}{4}$; (5) none of the above.

10. Two cities are 200 miles apart. A car travels from the first to the second at 50 miles per hour and returns at 40 miles per hour. The average speed for the round trip is closest to:

(1) 43 mph ; (2) 44 mph ; (3) 45 mph ; (4) 46 mph ;

(5) none of the above.

11. The units digit of a two digit number exceeds the tens digit by 4. If the digits are reversed, the square of the given number exceeds 4 times the new number by 21. The sum of the digits of the given number is:

(1) 8; (2) 10; (3) 12; (4) 14; (5) none of the above.

12. The fifth term in the expansion of $(a^3b^2 - a^{-3}b^{-2})^{12}$ is:

(1) $495\ a^{12}b^8$; (2) $-495\ a^{12}b^8$; (3) $495\ a^8b^{12}$;

(4) $-495\ a^8b^{12}$; (5) none of the above.

13. The expression

$$3 + \sqrt{3} + \frac{1}{3 - \sqrt{3}} + \frac{1}{3 + \sqrt{3}}$$ equals:

(1) $4 - \sqrt{3}$; (2) $4 + \sqrt{3}$; (3) $\sqrt{3}$; (4) $-\sqrt{3}$;

(5) $8 + \sqrt{3}$.

14. The equation $\frac{3x - 6}{x - 1} + 6 = \frac{-3}{x - 1}$ has:

(1) infinitely many roots; (2) one root; (3) two roots;

(4) no roots; (5) four roots.

15. The number .189189 . . . can be written as a fraction. When reduced to lowest terms the sum of the numerator and denominator of this fraction is:

(1) 44; (2) 132; (3) 1188; (4) 67; (5) none of the above.

16. The value of $\log_3 \frac{27 \times 3^a}{81}$ is equal to:

(1) $\frac{3}{4} + a$; (2) $\frac{3}{4} + \frac{a}{4}$; (3) $-(1 - a)$; (4) $1 - a$;

(5) none of the above.

17. What is the smallest integer which, when divided by 2 or 3 or 4, will leave a remainder of 1?

(1) 13; (2) 25; (3) 37; (4) 11; (5) none of the above.

18. In the Binary Numeration System, counting is as follows: 1, 10, 11, 100, 101, 110, 111, ... The numeral in the Binary Numeration System representing sixty:

(1) has seven digits.
(2) begins with "1" and ends with "1".
(3) has six digits.
(4) has five "1's".
(5) none of the above.

19. Let the operation $*$ be defined for the set of non-negative integers by:

$$a * b = |a - b|.$$

Which of the following statements is false?

(1) $5 * 3 = 2$

(2) The set of non-negative integers is closed with respect to $*$.

(3) The operation $*$ is commutative for all non-negative integers.

(4) The operation $*$ is associative for all non-negative integers.

(5) The identity element is 0.

20. Three pennies are tossed. What is the probability of obtaining at least <u>one</u> head?

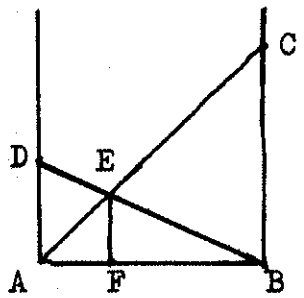
(1) 1/2 ; (2) 2/3 ; (3) 1/8 ; (4) 3/4 ;

(5) none of the above.

21. A box contains two white and three red balls. Two balls are drawn in succession without replacement. What is the probability that both are white?

(1) .1 ; (2) .65 ; (3) .4 ; (4) .15 ; (5) .6

22. If AB = 24, AC = 40, and BD = 25, where $\overline{AD}$, $\overline{AB}$, $\overline{BC}$ are in a plane and $\overline{BC}$ is perpendicular to $\overline{AB}$ and $\overline{AD}$ is perpendicular to $\overline{AB}$. Find EF.



(1) 6 ; (2) 13/3 ; (3) 15/2 ;

(4) 7 ; (5) none of the above.

23. The sides of a triangle are of lengths 7 in., 24 in., and 25 in. A perpendicular is dropped from the vertex opposite the side of length 25 in. to that side. What is the area of the smaller of the two triangles formed by the perpendicular?

(1) $\frac{4116}{625}$ sq. in. ; (2) $\frac{4096}{625}$ sq. in. ; (3) $\frac{4212}{625}$ sq. in. ;

(4) $\frac{4118}{625}$ sq. in. ; (5) none of the above.

24. Two circles, the lengths of whose radii are 3 inches and 8 inches, have their centers 13 inches apart. What is the length of the common external tangent?

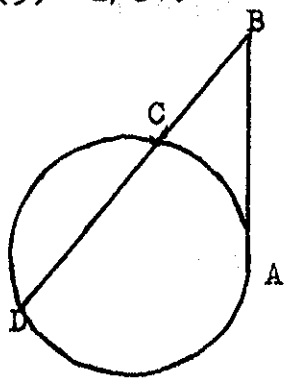
(1) 10 in. ; (2) 11 in. ; (3) 12 in. ; (4) 13 in. ;

(5) 14 in.

25. The midpoints of the sides of a triangle are joined by line segments forming smaller triangles. Each of the midpoints of the sides of the smaller triangles are joined by line segments forming still smaller triangles. If this process is performed three times, the area of each of the smallest triangles is what part of the area of the original triangle?

(1) 1/3; (2) 1/8; (3) 1/16; (4) 1/48; (5) 1/64.

26. A tangent $\overline{AB}$ to the circle is 8 in. A secant $\overline{BD}$ is drawn so as to form a chord $\overline{CD}$, which is 12 inches long. The least possible length of the radius of this circle is:



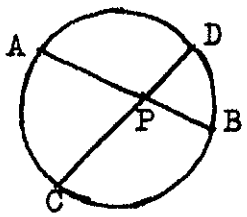
(1) 3 in.; (2) 4 in.; (3) 5 in.;

(4) 6 in.; (5) 8 in.

27. Given a circle with chords $\overline{AB}$ and $\overline{CD}$ intersecting at P:

Which of the statements is always true:

(1) AP = CP and DP = PB
(2) AP + PB = CP + PD
(3) AP + PD = CP + PB
(4) AP × PD = CP × PB
(5) CP × PD = AP × PB



28. Three circles of radii 7 inches are tangent externally to each other. The centers of the circles are A, B, and C. The area of the portion of the triangle ABC not included within the circles is: (use $\pi = \frac{22}{7}$)

(1) $49\sqrt{3} - 77$ sq. in. ; (2) $51\sqrt{3} - 85$ sq. in. ;

(3) $48\sqrt{3} - 80$ sq. in. ; (4) $42\sqrt{3} - 66$ sq. in. ;

(5) none of the above.

29. One end of a rope 63 ft. long is tied to a cow and the other end is tied to the lower corner of a barn whose dimensions are 35 ft. by 28 ft. If the movement of the cow is restricted only by the rope and the barn, over how many square feet of area can the cow graze? Use $\frac{22}{7}$ as an approximation for π.

(1) 10934; (2) 10226; (3) 10471; (4) 10016; (5) 10328.

30. Consider the statement: "If all men are free, then some men are good." The contrapositive of this statement is:

(1) If all men are bad, then some men are not free.
(2) If some men are not free, then all men are bad.
(3) If some men are bad, then all men are not free.
(4) If all men are bad, then all men are free.
(5) None of the above.

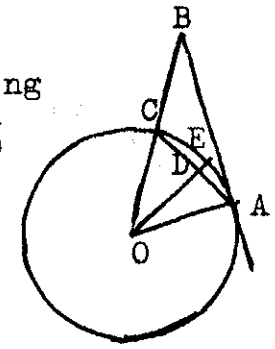
31. The total number of diagonals of a hexagon is:

(1) 6; (2) 7; (3) 8; (4) 9; (5) 10.

32. The sides of a rectangle are 9 in. and 16 in. The side of a square having the same area is:

(1) 10 in.; (2) 12 in.; (3) 11 in.; (4) 13 in.; (5) 14 in.

33. The radius $\overline{OA}$ of the circle is 7 units long. The tangent $\overline{AB}$ to the circle is 24 units long. O is joined to B by a line segment intersecting the circle at C, and the chord $\overline{CA}$ is drawn. A perpendicular to $\overline{CA}$ from O intersects $\overline{CA}$ at D and the circle at E. Find DE:



(1) 3/5; (2) 1; (3) 8/5
(4) 7/5; (5) 4/5.

34. Which of the following is not correct for the value of x in the equation?

$$\sin 6x + \sin 2x = 0$$

(1) 0° ; (2) 30° ; (3) 45° ; (4) 90° ; (5) 135°.

35. Two angles of a triangle are 30° and 45°, respectively. The length of the side opposite the 30° angle is 10. What is the length of the side opposite the 45° angle?

(1) $10\sqrt{2}$; (2) $10\sqrt{3}$; (3) 15; (4) 12; (5) $8\sqrt{6}$.

36. In a triangle, side a is 6 inches long, side b is 10 inches long, and angle C is 120°. Which of the following is the correct length of side c?

(1) 12; (2) 14; (3) 15; (4) 16; (5) 18.

37. Which of the following is an identity?

(1) $\sin 3x = 3 \sin x \cos x$

(2) $\sin^4 x + \cos^4 x = 1$

(3) $\tan x/2 = \dfrac{\sin x}{1 + \cos x}$

(4) $\tan x \cot x = \cos x$

(5) $\tan 3\theta = \dfrac{3 \tan \theta}{1 - \tan^3 \theta}$

38. A cylindrical oil tank 7 feet in radius and 10 feet long has an axis that is horizontal. The surface of the oil in the tank is in the shape of a rectangle $7\sqrt{2}$ feet wide and 10 feet long, and the surface lies below the axis of the cylinder. Assuming $\pi = \frac{22}{7}$, and that there are 7.5 gal. to the cubic foot, determine the number of gallons in the tank.

(1) 1020 gal.; (2) 1030 gal.; (3) 1040 gal.;

(4) 1050 gal.; (5) 1060 gal.

39. Assume the earth is a sphere such that the length of its radius is 4000 miles. To the nearest 500 square miles, determine the area one may sight from an observation tower which is .1 mile high.

(1) 2000 sq. miles; (2) 2500 sq. miles; (3) 3000 sq. miles;

(4) 3500 sq. miles; (5) 4000 sq. miles.

40. Assume the earth is a sphere whose great circle has a circumference of 25,000 miles. The latitude of Johnsboro is 30°12' N. If Hawkinsville is 300 miles due north of Johnsboro, what is the latitude of Hawkinsville to the closest minute?

(1) 36°16' (2) 34° 31' (3) 33° 57'

(4) 38°11' (5) 34° 10'