TMTA CONTEST – CALCULUS AND ADVANCED TOPICS

1. Suppose $Q = te^{-bt}$, with b a positive constant. For which values of t will Q increase?

   a) $t < b$ b) $t > b$ c) $t > \frac{1}{b}$ d) $t < \frac{1}{b}$ e) $t > b^2$

2. Find the finite area bounded by the graphs of $y = x^4$ and $y = x^2$.

   a) 0 b) 2/15 c) –2/15 d) 4/15 e)-4/15

3. A six-sided die has faces labeled 1 through 6. It is weighted so that a three is three times as likely to be rolled as a one; a three and a six are equally likely; and a one, a two, a four, and a five are equally likely. What is the probability of rolling a three?

   a) 1/6 b) 1/3 c) 2/3 d) 3/10 e) 2/5

4. At how many points does the graph of $g(x) = .11x^3 + .33x^2 - x + .66$ cross or touch the x-axis?

   a) 4 b) 0 c) 3 d) 2 e) 1

5. Consider a sports car which accelerates from 0 ft/sec to 88 ft/sec in 5 seconds (88 ft/sec = 60 mph). The car's velocity is given in the table below.

T	0	1	2	3	4	5
V(t)	0	30	52	68	80	88

   Find upper and lower bounds for the distance (in feet) the car travels in 5 seconds. Upper bound; Lower bound =

   a) 230; 185 b) 318; 295 c) 271; 258 d) 346; 318 e) 318; 230

6. Teresa is a 75% free throw shooter. If she shoots 5 free throws independently what is the approximate probability that she makes fewer than 5?

   a) 0.75 b) 0.2373 c) 0.7627 d) 1 e) 0.0146

7. I have a set of numerical data with one outlier. The mean $\bar{x}$ is 103.69 and the median is 101.37. What will happen to the mean $\bar{x}$ and the standard deviation s of the set if the outlier is deleted?

a) $\bar{x}$ will increase; s will increase
b) $\bar{x}$ will increase; s will decrease
c) $\bar{x}$ will decrease; s will increase
d) $\bar{x}$ will decrease; s will decrease
e) There is not enough information

8. An area of 225 square feet is to be enclosed by 2 non-overlapping squares whose sides are in the ratio of 3:4. Find the length of the sides of the smaller square.

a) 3 feet b) 12 feet c) 6 feet d) 15 feet e) 9 feet

9. If $(x+2y)^{20}$ is put into its expanded form, what number multiplies the variables of the middle term?

a) 189,190,144
b) 184,756
c) 343,982,080
d) 167,960
e) 171,991,040

10. Find the volume of the solid obtained by rotating the finite area bounded by $y=x^2$ and $y=x^4$ about the x-axis.

a) $8\pi/45$ b) $4\pi/45$ c) $8\pi/15$ d) $16\pi/15$ e) $4\pi/15$

11. What is the maximum value of the derivative of $f(x)=-x^3+6x^2-9x+1$?

a) 2 b)-3 c) 1 d) 3 e) –2

12. A sandbag is released from a balloon floating at 1000 meters above the ground; $h(t)$ is its height above the ground in meters and t is the time since its release in seconds. The sand bag falls freely except for air resistance. $h(t)$ satisfies the differential equation $h''(t)=-9.8-\frac{1}{10}h'(t)$. Which of the following is an expression for $h(t)$?

a) $1980-98t-980e^{-t/10}$
b) $1000-50t$
c) $1000-100e^{-t/10}-98t$
d) $-4.9t^2+1000$
e) $20+98t+980e^{-t/10}$

13. A dinghy is pulled toward a dock by a rope attached to the bow that goes through a ring on the dock 3ft above the bow. If the rope is hauled in at the rate of 1/2 ft/sec, how fast is the boat approaching the dock when 5ft of rope connects the boat to the dock?

a) 1 ft/sec b) 1/2 ft/sec c) 3/4 ft/sec
d) 5/8 ft/sec e) 4/9 ft/sec

14. Evaluate $\lim_{x\to 9} \frac{x(\sqrt{x}-3)}{x-9}$.

a) 1/3 b) 3/2 c) 1 d) 1/2 e) Does not exist

15. If $f(x) = .08x^4 - .5x^2 + .37x$, at how many points is the slope of f equal to 1?

a) 3 b) 2 c) 1 d) 0 e) infinitely many

16. Water is poured into a conical paper cup at the rate of 2/3 cubic inches per second. If the cup is 6 inches tall and the top of the cup has a radius of 2 inches, how fast does the water level rise when the water is 4 inches deep?

a) $\frac{3}{8\pi}$ in/sec b) $\frac{3\pi}{8}$ in/sec c) $\frac{8\pi}{3}$ in/sec
d) $\frac{8}{3\pi}$ in/sec e) $\frac{2\pi}{3}$ in/sec

17. Suppose 25% of all super-heroes are born with the gene that causes x-ray vision, and that the test for the gene is only 90% accurate. Given that Calculus Boy is a super-hero who tests positive for the gene, what is the probability that he actually has the gene that causes x-ray vision?

a) 0.225 b) 0.25 c) 0.90 d) 0.75 e) 0.10

18. Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |x| \sin x \, dx$.

a) $\frac{\sqrt{2}}{2}\left(1-\frac{\pi}{4}\right)$ b) $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}-1\right)$ c) 0
d) $-\frac{\pi\sqrt{2}}{4}$ e) $\frac{\pi\sqrt{2}}{4}$

19. Find the approximate volume of the solid bounded by the graphs of $y = x$, $y = x^2/4$, $z = 2$, and $z = -2$.

a) 21.33 b) 53.33 c) 5.33 d) 10.67 e) 26.67

20. If y is defined implicitly as a function of x by $x^{\ln y} = 3$, find dy/dx at the point where $x = \mathrm{e}$.

a) ln 9 b) 0 c) $\frac{-\ln 27}{\mathrm{e}}$ d) $\frac{\ln 27}{\mathrm{e}}$ e) $\frac{\ln 9}{\mathrm{e}}$

21. What is the equation of the axis of symmetry of the parabola passing through the points (0, -1), (1, 3) and (-3, -1), if the axis of symmetry is vertical?

a) $x = 1.5$ b) $x = -1.5$ c) $x = 1$ d) $x = -1$ e) $x = 3$

22. Thirty-six students took the ACT, with a mean score of 25.5. The boys had a mean score of 23.5, while the girls had a mean score of 28. How many girls were in the group?

a) 20 b) 18 c) 16 d) 14
e) There is not enough information

23. If $f(x) = \prod_{n=1}^{10}(x+n) = (x+1)(x+2)\cdot\ldots\cdot(x+n)$, then which of the following is the derivative of $f(x)$ evaluated at $x = 0$?

a) 10,500,000 b) 10,523,472 c) 11,123,593
d) 12,321,496 e) 10,628,640

24. There are 100 members of the senate, 2 from each state. If a committee of 5 is selected at random, how many ways can the committee be formed if no state may be represented more than once?

a) 2,118,760 b) 75,287,520 c) 4,950
d) 67,800,320 e) 254,251,200

25. A river 1 kilometer wide has a constant current of 5 kilometers per hour. At approximately what angle to the shore should a person head a boat that is capable of maintaining a speed of 15 kilometers per hour in order to reach a point directly opposite?
a) 60^o b) 70.5^o c) 72^o d) 76.5^o e) 81^o

26. Find the equation of the tangent to the curve $y = e^x$ which passes through the origin.

a) $y = e + x$ b) $y = x$ c) $y = ex$ d) $y = 2ex$ e) $y = x/e$

27. Compute $\int_{-\infty}^{\infty} \frac{x^3}{(x^4+1)^2} dx$.

a) –10 b) 0 c) 10 d) π e) $\pi/4$

28. If m and n are natural numbers, and $4m - 5n = 1$, what is the greatest common divisor of m and n?

a) 4 b) 5 c) 20 d) 1
e) It cannot be determined from the given information

29. Assume that a computation using method A takes $8n^2$ seconds, where $n \in \mathbf{N}$ represents the size of the input. Assume that method B performs the same computation in $64 n \log_2 n$ seconds. Which is the largest interval for n where A performs faster than B ?

a) $n \geq 44$ b) $n \geq 32$ c) $2 \leq n \leq 43$ d) $2 \leq n \leq 64$ e) $1 \leq n \leq 32$

30. A probability distribution has a mean of 23.5 and a standard deviation of 5.3. These values are scaled so that the new mean is 150 and the new standard deviation is 20. What value in the original distribution would result in a scaled value of 135?

a) 17 b) 17.595 c) 18 d) 19.255 e) 19.525

31. If $f(x) = x^3$ and $L(x)$ is the tangent line approximation to f at the point where $x = 1$, what is the average value of the error $|f(x) - L(x)|$ over the interval $[0,2]$?

a) 0 b) 0.5 c) 1 d) 1.5 e) 2

32. What is the derivative of $f(x) = x^x$, $x > 0$ evaluated at $x = e$?

a) $2e^e$ b) e^e c) $1 + e^e$ d) $(\ln e)e^e$ e) e^{e+1}

33. Given the curve $y = ax^3 + bx^2 + cx + d$, $a \neq 0$, find the relationship between the parameters a, b and c that will ensure that the curve has exactly one critical point.

a) $b^2 = 3ac$ b) $a = 3bc$ c) $c = 2a^2b$ d) $a = b^2c$ e) $b = a^2c$

34. The matrix equation $\begin{bmatrix} 2 & 0 \\ -10 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} x \\ y \end{bmatrix}$ where x, y and a are real numbers has more than one solution. Which of the following is a possible value for a?

a) 2 b) $2+\sqrt{3}$ c) 3 d) –2 e) $2-\sqrt{3}$

35. What is $\lim_{x \to 0} x^x$?

a) 0 b) 1 c) e d) 1/e e) Does not exist

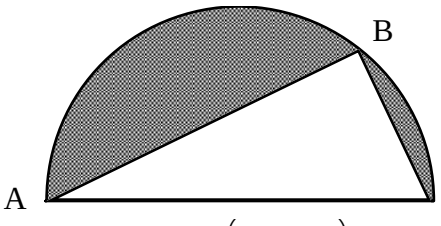
36. A person deposits $500 into a savings account at the end of every month for 4 years at 6% annual rate compounded monthly. How much interest will be earned during the 4 years?

a) $1440 b) $1480.27 c) $2024.39 d) $3048.92 e) $4098.46

37. Evaluate $\int_1^4 e^{\sqrt{x}} dx$.

a) e^4 b) $6e^4$ c) $\frac{e^2}{3} - \frac{e}{2}$ d) $e^2 - e$ e) $2e^2$

38. Find the area of the shaded region enclosed in a semicircle of diameter 8 units and $\angle BAC = 30^\circ$.



a) $8\left(2\sqrt{3} - \pi\right)$ b) $16\left(\sqrt{3} - \frac{\pi}{2}\right)$ c) $8\left(\pi - \sqrt{3}\right)$

d) $12\left(\pi - \sqrt{2}\right)$ e) Cannot be determined

39. $\Theta(g(n))$ is defined as an asymptotic tight bound of a function $f(n)$ if there exist positive constants c_1, c_2 and n_0, where $n_0 \in \mathbf{N}$ such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$. If $f(n) = 4n^2 - 6n$, which of the following values of c_1, c_2 and n_0 could be used to show $\Theta(n^2)$ is an asymptotic tight bound of $f(n)$?

a) $c_1 = 4$, $c_2 = 2$, $n_0 = 2$
b) $c_1 = 1$, $c_2 = 4$, $n_0 = 2$
c) $c_1 = 1$, $c_2 = 3$, $n_0 = 2$
d) $c_1 = 0$, $c_2 = 4$, $n_0 = 1$
e) $c_1 = 2$, $c_2 = 4$, $n_0 = 2$

40. The Lambert W-function $W(x)$ is defined to be the inverse of the function $f(x) = x\mathrm{e}^x$, $x \ge -1$, find $\int_0^{\mathrm{e}} W(x)\,dx$.

a) 1 b) e c) e+1 d) e-1 e) 0

Extra Problems

A. A plane is down and is presumed to have equal probability of going down in any of three regions. If the plane is down in region 1, the probability it will be found in a search is 0.95. Similarly, the probabilities for regions 2 and 3 are 0.90 and 0.80 respectively. Which of the following is the closest to the conditional probability that the plane is in region 2, given that a search of region 1 was unsuccessful?

a) 0.49 b) 1 c) 0.58 d) 0.34 e) 0.38

B. Find the area of the region between $y^2 = 2x - 5$ and $y = x - 4$.

a) 20 b) 20/3 c) 16 d) 16/3 e) 25/3

Solutions

1	D
2	D
3	D
4	E
5	E
6	C
7	E
8	E
9	A
10	A
11	D
12	A
13	D
14	B
15	A
16	A
17	D
18	C
19	D
20	C
21	B
22	C
23	E
24	D
25	B
26	C
27	B
28	D
29	C
30	E
31	C
32	A
33	A
34	A
35	E
36	D
37	E
38	C
39	B
40	D

A	A
B	D