TENNESSEE MATHEMATICS TEACHERS ASSOCIATION

SIXTY-FIFTH ANNUAL MATHEMATICS CONTEST

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Calculus and Advanced Topics

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Scoring Formula: $4 \times$ (Number Right) $-$ (Number Wrong) $+ 40$

Directions:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

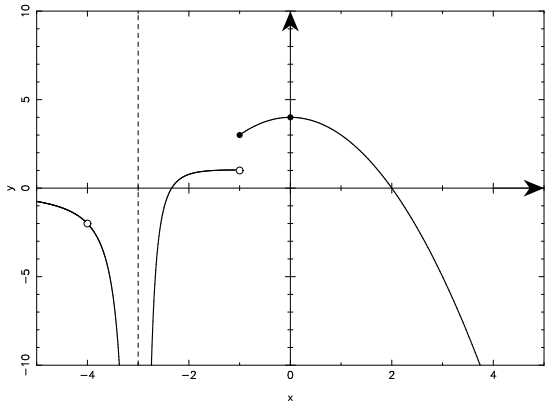
This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

TMTA Calculus and Advanced Topics Exam

1. The graph of a function $f(x)$ is represented in the figure below. Does this function have a removable discontinuity, and if yes, at what value of x?



a. The function $f(x)$ does not have a removable discontinuity.

b. yes, at $x = 0$

c. yes, at $x = -1$

d. yes, at $x = -3$

e. yes, at $x = -4$

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2. What is the value of the following limit?

$$\lim_{x \to 7} \frac{2x^2 - 13x - 7}{x - 7}$$

a. The limit does not exist.

b. 15

c. 8

d. $\frac{1}{15}$

e. 0

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3. Let $f(x) = \begin{cases} 5x^2 - 3x, & x < 1 \\ 3, & x = 1 \\ 7x - 5, & x > 1 \end{cases}$.
What is value of $\lim_{x \to 1} f(x)$?

a. The limit does not exist.

b. 3

c. 2

d. 1

e. 0

4. Which statement is NOT true?

   a. If $f(x)$ is differentiable at a point x, it is continuous at this point.

   b. If you integrate $f(x)$ to get an antiderivative, and then differentiate the antiderivative, you get the original function $f(x)$.

   c. If the tangent line to the graph of a continuous function $f(x)$ at $x = c$ is vertical, $f(x)$ is not differentiable at $x = c$.

   d. If $f(x)$ is not continuous at a point x, it is not differentiable at this point.

   e. If $f(x)$ is continuous at a point x, it is differentiable at this point.

5. What is an equation of the tangent line to the graph of $f(x) = \ln\left(\frac{\pi}{x}\right)$ at the point $(\pi, 0)$?

   a. $y = -\frac{1}{x} + \pi$

   b. $y = -\frac{x}{\pi} + \pi$

   c. $y = -\frac{\pi}{x} + 1$

   d. $y = -\frac{x}{\pi} + 1$

   e. $y = -\frac{\pi}{x} + \pi$

6. Let $f(x) = \frac{1 + 2x}{1 - 4x}$. What is $f'(0)$?

   a. 0

   b. $\frac{1}{2}$

   c. 2

   d. 4

   e. 6

7. Let $f(x) = \dfrac{1 + 9x - \sqrt{x}}{x^2}$. What is $f'(1)$?

   a. -9.5

   b. -6

   c. 6.5

   d. 8

   e. 9

8. Let $f(x) = 2x \ln x$. What is $f^{(2)}(e)$?

   a. 0

   b. $\dfrac{2}{e}$

   c. 2

   d. 4

   e. $2e$

9. Let $f(x) = e^{\cos(2x+5)}$. What is $f'(1)$?

   a. $2e^{\sin(7)}$

   b. $-2e^{\sin(7)} \sin(7)$

   c. $2e^{\cos(7)} \sin(7)$

   d. $-2e^{\cos(7)} \sin(7)$

   e. $2e^{\sin(7)} \cos(7)$

10. Which of the following functions is infinitely differentiable (has all derivatives $f^{(n)}(x)$ with $n \in \mathbb{N}$) on $(-\infty, \infty)$?

    a. A polynomial function

    b. A rational function

    c. $f(x) = x^{-n}$ with $n \in \mathbb{N}$

    d. $f(x) = \tan x$

    e. $f(x) = \ln x$

11. What is the solution set to the equation: $\ln\left(\frac{x-7}{x+7}\right) + \ln(x^2 - 49) = 0$?

    a. $\{0\}$
    b. $\{1\}$
    c. $\{4\}$
    d. $\{8\}$
    e. $\{16\}$

12. If $f'(x) = g'(x)$ on the interval $[-20, 20]$, which of the following statements is NOT correct?

    a. The instantaneous rate of change of $f(x)$ at $x = 0$ is the same as that of $g(x)$ at $x = 0$.
    b. The average rate of change of $f(x)$ on the interval $[-10, 5]$ is equal to the average rate of change of $g(x)$ on the same interval.
    c. $\int_1^{10} f(x)dx = \int_1^{10} g(x)dx$
    d. Both functions are integrable on the interval $[-20, 20]$.
    e. The tangent line to to graph of $g(x)$ at $x = 0$ may intersect the graph of $g(x)$ at more than one point.

13. If $F(x) = \int_1^{5x^2} \frac{t}{t^2 + 20} dt$, what is $F'(2)$?

    a. 0
    b. $\frac{x}{x^2 + 20}$
    c. $\frac{1}{20}$
    d. $\frac{1}{21}$
    e. $\frac{20}{21}$

14. To evaluate an integral, you decided to use the trigonometric substitution $x = 8\tan\theta$, and arrived at an intermediate result of $\sin\theta + C$ that you have to express in terms of x. What is the correct expression for $\sin\theta$ in terms of x in this case?

    a. $\dfrac{x}{8}$

    b. $\dfrac{\sqrt{64-x^2}}{x}$

    c. $\dfrac{x}{\sqrt{64-x^2}}$

    d. $\dfrac{x}{\sqrt{64+x^2}}$

    e. $\dfrac{8}{\sqrt{64+x^2}}$

15. Consider the following limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt[3]{2+\frac{3i}{n}}.$$

    The above limit is equal to which of the following definite integrals?

    a. $\displaystyle\int_2^5 \sqrt[3]{x}\,dx$

    b. $\displaystyle\int_0^3 3\sqrt[3]{x}\,dx$

    c. $\displaystyle\int_0^3 3\sqrt[3]{x+2}\,dx$

    d. $\displaystyle\int_2^3 \sqrt[3]{x+2}\,dx$

    e. $\displaystyle\int_2^5 3\sqrt[3]{3x+2}\,dx$

16. Evaluate $\displaystyle\int_0^3 |x^2-4x+3|dx$.

    a. 3

    b. $\dfrac{8}{3}$

    c. 2

    d. $\dfrac{3}{2}$

    e. 0

17. The most appropriate method to solve the integral $\int \tan^{-1} x \, dx$ is which of the following?

    a. A trigonometric substitution of $x = f(\theta)$
    b. A substitution of $u = f(x)$
    c. A trigonometric integrals method
    d. Integration by parts
    e. Partial fraction decomposition

18. Find all relative extrema of the function $f(x) = x - 4\sqrt[3]{x}$.

    a. DNE
    b. $(1, -3)$, $(-1, 3)$, and $(0, 0)$
    c. $(1, -3)$ and $(0, 0)$
    d. $(1, -3)$ and $(-1, 3)$
    e. $(1, -3)$

19. How many of the following statements are true?

    – For any continuous function $f(x)$, left endpoint L_n estimate of the area under the curve over $[a, b]$ is always an under-estimate of this area.
    – To solve the integral $\int \frac{x^2 + 2x - 1}{\sqrt[3]{x^3 + 3x^2 - 3x}} dx$ by the method of substitution, you should set the new variable u to $u = x^2 + 2x - 1$.
    – The integral $\int \frac{-x^2 + 12x + 9}{7x^2 + 3} dx$ is improper.
    – The area defined by an improper integral is unbounded.

    a. 0
    b. 1
    c. 2
    d. 3
    e. 4

20. Which of the following definite integrals can be used to compute the volume of a solid obtained by rotating the region bounded by the curve $y = x^3 + 3$ and the lines $x = 0$, $x = 1$, and $y = 0$ about the line $x = -2$?

    a. $\pi \int_0^1 (x^3 + 3)^2 \, dx$

    b. $\pi \int_0^1 (x^3 + 5)^2 \, dx$

    c. $\pi \int_0^1 \Big((x^3 + 1)^2 - 4\Big) \, dx$

    d. $\pi \int_0^1 \Big((x^3 + 3)^2 - 4\Big) \, dx$

    e. $\pi \int_0^1 \Big((x^3 + 5)^2 - 4\Big) \, dx$

21. The average value of $f(x)$ on $[-1, 8]$ is 3. What is the value of the integral $\int_{-1}^{8} f(x)dx$?

    a. $\frac{1}{9}$

    b. $\frac{1}{3}$

    c. 3

    d. 9

    e. 27

22. Given $f(x) = e^{ax^2+bx+1}$. If $f(1) = f(0) = f'(0)$, what is the value of a?

    a. $a = 0$

    b. $a = 1$

    c. $a = -1$

    d. $a = 2$

    e. $a = -2$

23. The position function of a fly is given by the formula $s(t) = 8 - 2t + 24t^2 - 0.3t^5$. At what time t does the fly's speed reach its maximum?

    a. $t = 2$

    b. $t = 3$

    c. $t = 3.5$

    d. $t = 4$

    e. $t = 2\sqrt{5}$

24. What is the half-life (to the nearest tenth of a year) of a certain radioactive material if after 2 years 99% of the initial amount remains?

    a. 67.2 years
    b. 88.1 years
    c. 102.7 years
    d. 137.9 years
    e. 192.3 years

25. What is the value of the following definite integral?
$\int_0^{\pi/12} \sec(2\theta)\tan(2\theta)d\theta$

    a. $\frac{\sqrt{3}}{3} - \frac{1}{2}$
    b. $\frac{2}{\sqrt{3}} - 1$
    c. $\sqrt{3} - \frac{1}{2}$
    d. $\sqrt{3} + \frac{1}{2}$
    e. none of these

26. What is the volume of a solid obtained by rotating the region bounded by $y = 2 - 2x^2$, $y = 2$, $x = 1$ about the line $y = 3$?

    a. 2
    b. 2π
    c. $\frac{32\pi}{15}$
    d. $\frac{31\pi}{16}$
    e. $\frac{3\pi}{4}$

27. What is the area enclosed between the graphs of $x = 1$, $y = e^x$, and $y = e^{-x}$?

    a. $e + \frac{1}{e} - 2$
    b. $e - \frac{1}{e}$
    c. $e - 2$
    d. $2 - \frac{1}{e}$
    e. none of these

28. Which of the following power series has the interval of convergence $[-1, 3)$?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2x^n}{4^n}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{4^n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$

e. none of these

29. Evaluate the definite integral $\displaystyle\int \sec^3 x \tan^3 x \, dx$.

a. $\dfrac{1}{5}\sec^5 x + \dfrac{1}{3}\sec^3 x + C$

b. $\dfrac{1}{5}\sec^5 x - \dfrac{1}{3}\sec^3 x + C$

c. $\dfrac{1}{4}\sec^4 x - \dfrac{1}{2}\sec^2 x + C$

d. $\dfrac{1}{4}\sec^4 x + \dfrac{1}{2}\sec^2 x + C$

e. $\dfrac{1}{16}\sec^4 x \tan^4 x + C$

30. On what intervals is the function $f(x) = \dfrac{x^2}{x^2+9}$ is concave upward?

a. $(-3, -\sqrt{3}) \cup (\sqrt{3}, 3)$

b. $(-3, 3)$

c. $(-\sqrt{3}, \sqrt{3})$

d. $(-\infty, -3) \cup (3, \infty)$

e. $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

31. What is the sum of the following geometric series?

$$\sum_{n=0}^{\infty}(-3)^{-n+1}$$

a. the series diverges because the common ratio $|r| > 1$

b. $S = -\dfrac{9}{4}$

c. $S = -\dfrac{3}{2}$

d. $S = \dfrac{3}{2}$

e. $S = \dfrac{9}{4}$

32. What comparison will show that the integral $\displaystyle\int_1^{\infty} \frac{dx}{x(2+\cos x)}$ diverges?

a. with $\dfrac{1}{2}\displaystyle\int_1^{\infty} \frac{dx}{x}$

b. with $\dfrac{1}{3}\displaystyle\int_1^{\infty} \frac{dx}{x}$

c. with $\displaystyle\int_1^{\infty} \frac{dx}{2+\cos x}$

d. with $\displaystyle\int_1^{\infty} \frac{dx}{x\cos x}$

e. the comparison test is not applicable in this case

33. What are the vertical and horizontal asymptotes for the following function?
$f(x) = \dfrac{2x^2+11}{x^2-9}$

a. horizontal asymptote $y = -\dfrac{11}{9}$, vertical asymptote $x = 3$

b. horizontal asymptote $y = 2$, vertical asymptotes $x = \pm 3$

c. horizontal asymptotes $y = \pm 3$, vertical asymptote $x = 2$

d. horizontal asymptote $y = \dfrac{1}{2}$, vertical asymptote $x = 3$

e. no horizontal asymptote, vertical asymptotes $x = \pm 3$

34. What is the minimum number of terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n}$ that are required to estimate the sum of the series with an error less than 0.0001?

a. 4
b. 5
c. 6
d. 7
e. 8

35. What is the general antiderivative of the function $f(x) = \frac{1}{1+\sqrt{x}}$?

a. $\ln(1+\sqrt{x}) + C$
b. $\frac{1+\sqrt{x}}{(1+\sqrt{x})^2} + C$
c. $\frac{1}{2\sqrt{x}(1+\sqrt{x})^2} + C$
d. $2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$
e. $-\frac{\sqrt{x}}{(1+\sqrt{x})^2} + C$

36. When integrating a rational function, you get a sum of elementary functions. Which of the following functions can you NOT get when integrating a rational function?

a. a radical
b. a rational function
c. a polynomial
d. an arctangent
e. a logarithm

37. There is a 50% chance that the queen carries the gene for hemophilia. If she is a carrier, then each prince has a 50% chance of having hemophilia. If the queen has three princes without the disease, what is the probability that the queen is the carrier rounded to the nearest thousandth?

a. 0.111
b. 0.125
c. 0.212
d. 0.375
e. 0.5

38. What is the exact value of $\sin 15°$?

a. $\dfrac{\sqrt{3}-\sqrt{2}}{2}$

b. $\dfrac{\sqrt{3}-\sqrt{2}}{4}$

c. $\dfrac{\sqrt{5}-\sqrt{3}}{2}$

d. $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

e. $\dfrac{\sqrt{6}-\sqrt{5}}{2}$

39. What is the value of $\sqrt{\dfrac{1}{2}+\sqrt{\dfrac{1}{2}+\sqrt{\dfrac{1}{2}+\sqrt{\dfrac{1}{2}+\cdots}}}}$?

a. $\sqrt{2}$

b. $\dfrac{1+\sqrt{3}}{2}$

c. $\dfrac{1+\sqrt{2}}{2}$

d. $\sqrt{3}$

e. $\dfrac{4-\sqrt{2}}{2}$

40. The sequence a_n is defined recursively as follows:

$$a_1 = 0 \text{ and } a_{n+1} = \frac{n}{n+1}(a_n + 1) \text{ for } n \geq 1.$$

What is the value of the term a_{1000}?

a. $\dfrac{1000^2}{999}$

b. $\dfrac{1000 \cdot 999}{2}$

c. $\dfrac{999 \cdot 998}{2}$

d. $\dfrac{999}{2}$

e. 500