**TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION**

# SIXTIETH ANNUAL MATHEMATICS CONTEST
# 2016

## Calculus and Advanced Topics


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Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

# TMTA Calculus and Advanced Topics

1. $\lim_{x\to 0} x\cot x$.

A) 0 B) 1 C) $\infty$ D) $\frac{\pi}{2}$ E) $\frac{\pi}{4}$

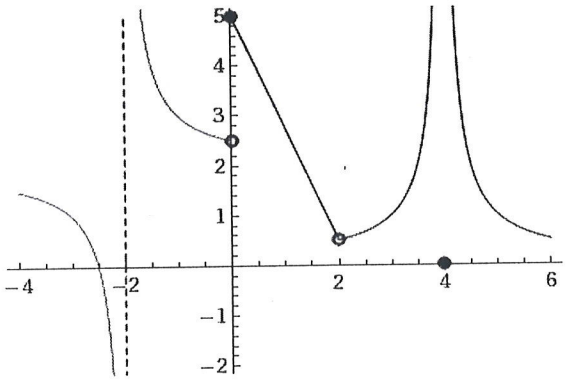
2. $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}$.

A) 1 B) $\infty$ C) $e^6$ D) $\frac{3}{2}$ E) $e^{\frac{2}{3}}$

3. If $f(4)=6$, $f'(4)=-3$, $g(4)=-4$, $g'(4)=3$, find $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x=4$.

A) $-1$ B) $-\frac{3}{8}$ C) $\frac{15}{8}$ D) $\frac{3}{8}$ E) 30

4. Given the graph of $f(x)$, which of the following statements is false.




A) $\lim_{x\to -2} f(x)$ does not exist.

B) $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ both exist.

C) $\lim_{x\to 2} f(x) = f(2)$.

D) $f(4)=0$, but $\lim_{x\to 4} f(x)$ does not exist.

E) $\lim_{x\to 0^+} f(x) = f(0)$

5. Find the absolute maximum of $y=\left|x^2-3x-4\right|$ on the interval $[0,5]$

A) 4 B) 5 C) 6 D) $6\frac{1}{4}$ E) $6\frac{3}{4}$

6. A tennis ball is dropped vertically from a height of 2 ft and begins bouncing. The height of each bounce is one-third the height of the previous bounce. What is the total distance the ball traveled before it stops on the ground.

A) $3\frac{2}{3}$ ft B) 4 ft C) $5\frac{1}{3}$ ft D) 6 ft E) 8 ft

7. Find the partial sum $S_n$ of the series $\sum_{k=1}^{n} a_k$, if $a_k = \ln\left(\frac{k^2}{k^2+2k+1}\right)$

A) $S_n = 2\ln(n)$ B) $S_n = \ln(n^2+1)$ C)

$S_n = -2\ln(n+1)$

D) $S_n = \ln\left(\frac{n}{n+1}\right)$ E) $S_n = (\ln(n+1))^2$

8. Find $a$ and $b$ such that f is differentiable everywhere.

$$f(x) = \begin{cases} ax^3, & x \le 4 \\ x^2 + b, & x > 4 \end{cases}$$

A) $a = \frac{1}{6}, b = -\frac{16}{3}$ B) $a = \frac{1}{2}, b = -\frac{1}{4}$

C) $a = \frac{1}{3}, b = \frac{7}{3}$ D) $a = -\frac{1}{6}, b = \frac{5}{6}$ E) $a = -\frac{2}{3}, b = -\frac{13}{6}$

9. Find the slope of the tangent line to the curve $x^2y + e^y = x$ at the point (1, 0).

A) 0 B) $\frac{1}{2}$ C) $\frac{3}{4}$ D) $\frac{1}{e}$ E) $\frac{1}{2e-1}$

10. Find the area of the region bounded between $y = \frac{1}{2}x$ and $y = x^2 - 2|x|$.

A) $\frac{32}{9}$ B) $\frac{13}{4}$ C) $\frac{27}{8}$ D) $\frac{31}{12}$ E) $\frac{19}{6}$

11. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt[3]{\frac{k}{n}} \cdot \frac{1}{n}$.

A) 0 B) $\frac{1}{3}$ C) $\frac{3}{4}$ D) 1 E) $\infty$

12. For what value of $t$ are vectors $\langle 8\sin(t), -2\cos(t), t\rangle$ and $\langle -\sin(t), 4\cos(t), \sqrt{t}\rangle$

orthogonal.

A) $t=0$ B) $t=\frac{\pi}{2}$ C) $t=\pi$ D) $t=\frac{3\pi}{4}$ E) $t=4$

13. Evaluate $\int_0^\infty te^{-st}dt$ for $s>0$.

A) 0 B) 1 C) $s$ D) $\frac{1}{s}$ E) $\frac{1}{s^2}$

14. Given a list of numbers $5,x,9,2$. For how many values of $x$ will the mean and median be equal?

A) 0 B) 1 C) 2 D) 3 E) 4

15. Solve the equation $x^{x^{x^{\cdots}}}=e$

A) $x=1$ B) $x=\frac{1}{e}$ C) $x=e^{\frac{1}{e}}$ D) $x=\frac{\sqrt{2}}{4}$ E) Does not exist

16. Which of the following is a solution to the differential equation $\frac{dy}{dx}-\frac{y}{x}=xe^x$, $y(1)=e-1$?

A) $xe^x-x$ B) $xe^x-1$ C) $e^x-x$ D) $\frac{e}{x}-1$ E) $\frac{e}{x^2}-x$

17. Convert the polar equation $r=\sec\theta\tan\theta$ to rectangular form.

A) $y=x$ B) $y=x^2$ C) $y=\sqrt{x}$ D) $y=\frac{1}{x}$ E) $x^2+y^2=1$

18. Given $x=e^t$, $y=t^3$, find $\frac{d^2y}{dx^2}$.

A) $\frac{6t}{e^t}$ B) $\frac{6t-3t^2}{e^t}$ C) $\frac{6t-3t^2}{e^{2t}}$ D) $\frac{3t^2}{e^{2t}}$ E) $\frac{6t}{e^{2t}}$

19. $\lim_{x\to 0}\frac{|x+2|-|x-2|}{2x}$.

A) $-1$ B) 0 C) 1 D) 2 E) Does not exist

20. Given matrix $A=\begin{bmatrix}0.4 & 0.6\\ 0.3 & 0.7\end{bmatrix}$. Find the determinant of matrix $P$, where $P=\lim_{n\to\infty}A^n$.

A) 0.1 B) 0.5 C) $-1$ D) $\frac{1}{3}$ E) 0

21. Determine the equation of the tangent line to the curve $\frac{x^2}{6}+\frac{y^2}{3}=1$ passing through the point $(-2,1)$.

A) $y=\frac{1}{2}x+2$ B) $y=\frac{2}{3}x+\frac{7}{3}$ C) $y=\frac{\sqrt{3}}{2}x+\sqrt{3}+1$

D) $y=x+3$ E) $y=1$

22. According to *Newton's Law of Cooling*, the rate of change of an object's temperature ($T$) is proportional to the difference between its temperature and the ambient temperature ($A$).

$$\frac{dT}{dt}=-k(T-A)$$

A can of soda at $40^\circ F$ is placed into a room where the temperature is $70^\circ F$. After 10 minutes the temperature of the soda is $50^\circ F$. What is the temperature of the soda 30 minutes after the soda was placed into the room?

A) $55^\circ F$ B) $58^\circ F$ C) $61^\circ F$ D) $64^\circ F$ E) $67^\circ F$

23. $x^{10}-x^5+3$ is divided by $x^2-1$. Find the remainder.

A) $-3$ B) $x+1$ C) $3x-2$ D) $-2x+3$ E) $-x+4$

24. Knowing $\lim_{n\to\infty} a_n=0$, and $a_{n+1}\le a_n$ which of following statements is true?

I. $\sum_{n=1}^{\infty} a_n$ converges. II. $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. III. $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges.

IV. $\sum_{n=1}^{\infty}\sqrt{a_n}$ converges. V. $\sum_{n=1}^{\infty}\frac{a_{n+1}}{a_n}$ converges.

A) I, III are true. B) Only II is true. C) II , III are true.

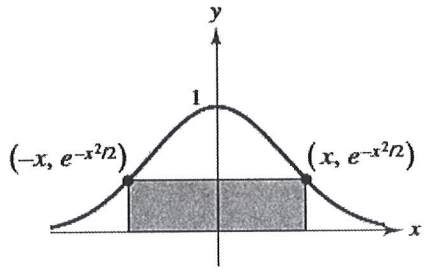
D) III, IV are true. E) IV, V are true.

25. Find the arc length of the graph of $y=\frac{x^2}{4}-\frac{\ln(x)}{2}$ on the interval $[1,e]$.

A) $\frac{e+1}{4}$ B) $\frac{2e-1}{4}$ C) $\frac{e^2-3}{2}$ D) $\frac{e^2+1}{4}$ E) $\frac{e^2-e}{4}$

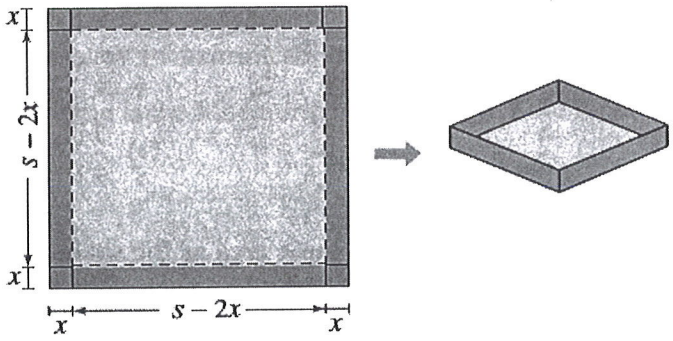
26. Consider the rectangle shown in the figure. Find the rate of change of the area when

$x = 3$ centimeters if $\frac{dx}{dt} = 4$ centimeters per minute. (Round your answer to two decimal places.)




A) $-0.71$ B) -0.66 C) $-0.58$ D) 0.23 E) 0.47

27. An open box of maximum volume is to be made from a square piece of material, $s = 12$ inches on a side, by cutting equal squares from the corners and turning up the sides (see figure).




Find the maximum volume.

A) 108 B) 121.5 C) 122.5 D) 128 E) 144

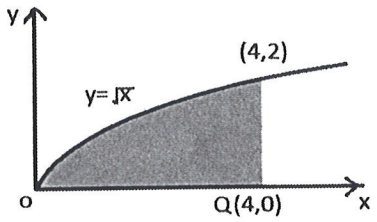
28. Which of the following is the value of the indefinite integral $\int \frac{x^2+3}{1-x^4}\,dx$ ?

A) $\ln(1-x^4)+C$ B) $\arctan\left(\ln\frac{1+x}{1-x}\right)+C$

C) $\arctan\left(\frac{1+x}{1-x}\right)+\ln(x)+C$ D) $\ln\left(\frac{1+x}{1-x}\right)+\arctan(x)+C$

E) $\ln(1-x^2)+\arctan(1+x^2)+C$

The graph below is for problem 29-31.




29. $P(x, y)$ is a point on the graph of $y = \sqrt{x}$. Find the value of $x$ such that the distance between $P$ and $Q(4, 0)$ is minimized.

A) $x = \frac{\sqrt{30}}{2}$ B) $x = 3\sqrt{2}$ C) $x = \frac{3\sqrt{5}}{2}$ D) $x = 2\sqrt{3}$ E) $x = \frac{7}{2}$

30. Find the ratio of the volumes of the solids formed by revolving the shaded region about the *x-axis* and *y-axis* respectively.

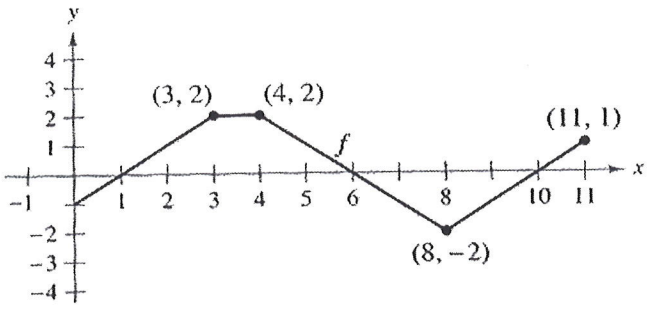
A) $\frac{3}{4}$ B) $\frac{5}{8}$ C) $\frac{5}{16}$ D) $\frac{11}{32}$ E) $\frac{27}{64}$

31. There exists a rectangle which has the same area as the shaded region with one side equals 4. Find the length of its diagonal. (Round your answer to one decimal place.)

A) 3.9 B) 4.2 C) 4.4 D) 4.7 E) 5.1

32. The graph of $f$ consists of line segments, as shown in the figure. Evaluate the definite integral

$\int_0^{11} f(x)dx$




A) $-\frac{1}{2}$ B) 0 C) 2 D) $\frac{5}{2}$ E) 11

33. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n^{n-1}(x-1)^n}{(n-1)!}$.

A) $[1-\frac{1}{e}, 1+\frac{1}{e})$ B) $[0,2)$ C) $[1,3)$

D) $[1-e, 1+e)$ E) $[-\frac{1}{2}, \frac{1}{2})$

34. Find $\frac{d}{dx}\int_0^{\frac{x^2}{2}} te^{-t}dt$.

A) $\frac{x^3}{2}e^{-\frac{x^2}{2}}$ B) $\frac{x^2}{2}e^{-\frac{x^2}{2}}$ C) $xe^{-\frac{x^2}{2}}$ D) $\frac{x^2}{2}e^{-x}$ E) $\frac{x^2}{2}e^{-\frac{x^3}{2}}$

35. Which of the following is the Maclaurin polynomial for the function $f(x) = \cos^2(x)$?

A) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n}}{(2n+1)!}$ B) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n-1} \cdot x^{2n}}{(2n)!}$

C) $1+\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n-1}}{(2n)!}$ D) $1+\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n-1} \cdot x^{2n}}{(2n)!}$

E) $\frac{1}{2}+\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n-1}}{(2n-1)!}$

36. Find the derivative of $f(x) = \arctan(\sqrt{x^2-1})$.

A) $f'(x) = \frac{1}{x^2}$ B) $f'(x) = \frac{x}{\sqrt{x^2-1}}$ C) $f'(x) = \frac{1}{x(x^2-1)}$

D) $f'(x) = \frac{1}{x\sqrt{x^2-1}}$ E) $f'(x) = \frac{\sqrt{x^2-1}}{x^2}$

37. Given $\sin(17^\circ) = a$, Express $\sin(56^\circ)$ in terms of $a$.

A) $1-\sqrt{1-a^2}$ B) $\sqrt{1-a^2}-a$ C) $\sqrt{1-a}-\sqrt{a}$ D) $1-a^2$ E) $1-2a^2$

38. Find $z$ such that $z^2 - z + 1 + i = 0$

A) $z = -i, i - 1$ B) $z = 1 + i, 1 - i$ C) $z = i, 1 - i$

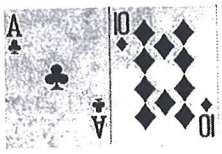
D) $z = 2i, 1 - 2i$ E) $z = -2i, 1 + i$

39. The solution set of the inequality $\frac{2x^2 - 5x + 2}{1 - x^3} \leq 0$ is

A) $(-\infty, \frac{1}{2}] \cup (1,2]$ B) $[\frac{1}{2}, 1) \cup [2, \infty)$ C) $[\frac{1}{2}, 1) \cup (1,2]$

D) $(-\infty, 1) \cup (2, \infty]$ E) $(-\infty, \frac{1}{2}] \cup [2, \infty)$

40. In the game of ***Blackjack***, face cards (kings, queens, and jacks) are counted as ten points. An ace is counted as either one point or eleven points. All other cards are counted as the numeric value shown on the card. Two cards are drawn from a standard deck of cards (52 cards). What is the probability of getting twenty or twenty-one points? (Round your answer to two decimal places)



A) 0.06 B) 0.11 C) 0.13 D) 0.15 E) 0.17

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## Calculus and Advanced Topics 2016

## ERRATA & NOTES

As you are taking the Calculus & Advanced Topics exam, please be aware of the following:

- Questions 5, 12, 20, and 26 are all split between two pages. Be sure to look on the next page for the continuation (and answer choices) for each of these.
- In question 7, answer choice C) is printed on the next line by itself instead of immediately following the "C)" marker.
- The graph on the top of page 8 should be used for THREE questions: 29, 30, and 31. (The instructions indicating this were unfortunately printed at the bottom of page 7).