FIFTY-EIGHTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Calculus and Advanced Topics 2014

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Scoring formula: 4R − W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the <u>best</u> answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase <u>completely</u>. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

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1) $\int \frac{3x^3 dx}{(x^4+5)^6} =$

a) $\frac{-12}{5(x^4+5)^5} + C$

b) $\frac{-3}{28(x^4+5)^7} + C$

c) $\frac{-3}{20(x^4+5)^5} + C$

d) $\frac{21x^4}{4(x^4+5)^7} + C$

e) $\frac{1}{5(x^4+5)^5} + C$

2) Estimate the value of $\int_0^1 e^{x^2} dx$ using four rectangles and a right-endpoint approximation.

a) ≈ 1.28

b) ≈ 1.71

c) ≈ 1.94

d) ≈ 5.1

e) ≈ 6.82

3) Which of the following is a true statement about $(x) = \frac{|x^2+x|}{1+x}$?

a) $\lim_{x\to+1^-} f(x) = 1$

b) $\lim_{x\to-1^+} f(x) = -1$

c) $\lim_{x\to-1^-} f(x) = 0$

d) $\lim_{x\to+1^+} f(x)$ does not exist

e) $\lim_{x\to 0} f(x)$ does not exist

4) Find $\frac{d^{2012}}{dx^{2012}}(2013 + 2012x + 2011x^2 + 2010x^3 + \cdots + 3x^{2010} + 2x^{2011} + x^{2012})$

a) $2012 + 4022x + 6030x^2 + \cdots + 4022x^{2010} + 2012x^{2011}$

b) 2012

c) 0

d) $2012!$

e) $2012! + 2011!\,x + 2012!\,x^2 + \cdots + 2!\,x^{2010} + 1!\,x^{2011} + 0!\,x^{2012}$

5) Where does $f(x) = \begin{cases} e^x - 1 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ \frac{1}{2-x} & \text{if } x > 1 \end{cases}$ fail to be continuous?

a) $x \le 0$

b) $x = 0$

c) $x = 1$

d) $x = 2$

e) f is continuous everywhere

6) $\int e^{3x}\cos(2x)dx =$

a) $\frac{1}{6}e^{3x}\sin(2x) + C$

b) $\frac{1}{2}e^{3x}\left(\sin(2x) + \frac{3}{2}\cos(2x)\right) + C$

c) $\frac{2}{13}e^{3x}\left(\sin(2x) + \frac{3}{2}\cos(2x)\right) + C$

d) $\frac{1}{2}e^{3x}\left(\sin(2x) + \frac{9}{4}\cos(2x)\right) + C$

e) $6e^{3x}\sin(2x) + C$

7) You plan to make and sell fancy hand-carved doorstops. When you sell x doorstops, the revenue in dollars is given by $r(x) = 6x$, and the cost (in dollars) is given by $c(x) = x^3 - 6x^2 + 15x$. How many doorstops should you sell in order to maximize your profit?

a) 6

b) $\sqrt{11} - 2$

c) 1

d) 0

e) As many as possible, because there is no maximum profit

8) On which interval(s) is $f(x) = ax^4 + bx^2$ increasing if $a < 0 < b$?

a) $\left(-\infty, -\sqrt{\frac{b}{2a}}\right] \cup \left[0, \sqrt{\frac{b}{2a}}\right]$

b) $[0, \infty)$

c) $\left[-\sqrt{\frac{-b}{2a}}, \sqrt{\frac{-b}{2a}}\right]$

d) $\left[-\sqrt{\frac{-b}{2a}}, 0\right] \cup \left[\sqrt{\frac{-b}{2a}}, \infty\right)$

e) $\left(-\infty, -\sqrt{\frac{-b}{2a}}\right] \cup \left[0, \sqrt{\frac{-b}{2a}}\right]$

9) On which interval(s) is $f(x) = \frac{1}{2}x - \frac{1}{4}\cos(2x)$ concave down?

a) $\bigcup_{n\in\mathbb{Z}}\left(\frac{(4n+1)\pi}{4}, \frac{(4n+3)\pi}{4}\right)$

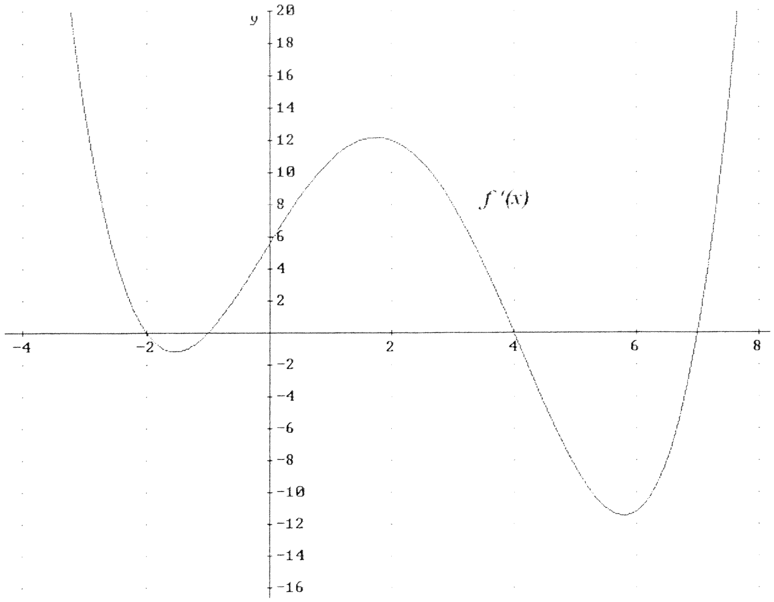
b) $\bigcup_{n\in\mathbb{Z}}\left(\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2}\right)$

c) $\bigcup_{n\in\mathbb{Z}}\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$

d) $\bigcup_{n\in\mathbb{Z}}\left(n\pi, \frac{(4n+2)\pi}{4}\right)$

e) $\bigcup_{n\in\mathbb{Z}}\left(\frac{(4n+3)\pi}{4}, \frac{(4n+5)\pi}{4}\right)$

10) The graph below is the graph of f', the first derivative of a function f. On which of the following intervals is $f(x)$ concave up?



a) $(-2,-1)$
b) $(4,6)$
c) $(2,4)$
d) $(6,7)$
e) $f(x)$ is never concave up

11) Jerk can be defined as the instantaneous rate of change in the acceleration of an object. If the position of a body moving along a coordinate line is given by $s(t) = \frac{25}{t+5}$, with s in meters and t in seconds, find the jerk at $t = 0$.
a) $2/5\ \mathrm{m}/s^3$
b) $-1\ \mathrm{m}/s^3$
c) $-2/625\ \mathrm{m}/s^3$
d) $-6/25\ \mathrm{m}/s^3$
e) $5\ \mathrm{m}/s^3$

12) $\lim_{x\to\infty} \frac{e^{3x^2}}{x^2} =$
a) Not equal to $-\infty$ or ∞, but does not exist.
b) $-\infty$
c) 0
d) e
e) ∞

13) If $f(x) = \sqrt{x-1}$ then the Mean Value Theorem asserts that there is a point in the interval [1,3] where the slope of the tangent line equals the slope of the secant line over that interval. At which point does this equality occur?

a) $x = 2$

b) $x = 1$

c) $x = \sqrt{2}$

d) $x = \frac{3}{2}$

e) the Mean Value Theorem does not apply to this function on the given interval

14) The absolute maximum of the function $f(x) = x^3 + x^2 - 8x + 5$ on the interval $[-1,2]$ occurs at

a) $x = \frac{4}{3}$

b) $x = 2$

c) $x = \frac{3}{2}$

d) $x = -2$

e) $x = -1$

15) The equation of the tangent line to $f(x) = 2\sin^2 x$ at $x = \pi/3$ is

a) $y = 2\sqrt{3}x - \frac{2\pi}{\sqrt{3}} + 3$

b) $y = \sqrt{3}x - \frac{\sqrt{3}\pi}{3} + \frac{3}{2}$

c) $y = \sqrt{3}x - \frac{\sqrt{3}\pi}{3} + \frac{1}{2}$

d) $y = 2\sqrt{3}x - \frac{\sqrt{3}\pi}{3} + \frac{1}{2}$

e) $y = \frac{\sqrt{3}}{2}x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

16) Find the value of $\cos(\csc^{-1} x)$.

a) $\frac{\sqrt{x^2-1}}{x}$

b) $\frac{x}{\sqrt{1-x^2}}$

c) $\sqrt{x^2-1}$

d) $\frac{1}{\sqrt{1-x^2}}$

e) $\frac{x}{\sqrt{x^2-1}}$

17) On which of the following intervals will the function $f(x) = |x^3 - 15x + 1| - 4$ have a root?

a) $[4,6]$
b) $[1,3]$
c) $[-7,-5]$
d) $[-3,-1]$
e) $[-1,0]$

18) $\frac{d}{dx}\left(\ln\sqrt{\frac{x^2+1}{(5x)^4}}\right) =$

a) $\sqrt{\frac{(5x)^4}{x^2+1}}$
b) $\frac{x}{x^2+1} - \frac{2}{x}$
c) $\frac{2(5x)^4}{x^2+1}$
d) $\frac{x}{x^2+1} - \frac{2}{125x^4}$
e) $\frac{1}{25\sqrt{2}x}$

19) The sum $\sum_{k=1}^{n} \frac{4k^3}{n^2}$ expressed in closed form is:

a) $4n$
b) $(n+1)^2$
c) $\frac{2(n+1)}{n}$
d) $2(n+1)$
e) $\frac{2(n+1)(2n+1)}{3n}$

20) A hot air balloon is rising at a speed of 3 ft/s and drops a sandbag when it reaches a height of 300 ft. What will the speed of the sandbag be when it hits the ground?

a) 138.44 ft/s
b) 64.8 ft/s
c) 76.77 ft/s
d) 70.7 ft/s
e) 32 ft/s

21) The area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is

a) 36π
b) $\frac{9\pi}{4}$
c) 6π
d) $\frac{4\pi}{9}$
e) $\frac{2\pi}{3}$

22) A round hole of radius s is drilled through the center of a sphere of radius r. What is the volume of the portion removed (i.e., the volume of the hole)?

a) $-\frac{4}{3}\pi\left((r^2 - s^2)^{\frac{3}{2}} - r^3\right)$
b) $\frac{4}{3}\pi r^3 - 2\pi s^2 r$
c) $2\pi s^2 r$
d) $4\pi\left(r^2 s - \frac{s^3}{3}\right)$
e) $-\frac{4}{3}s^{\frac{3}{2}}$

23) A ball is dropped from a height of 3 meters, and each time it hits the ground it bounces vertically to a height that is 3/5 of the height on the preceding bounce. If we assume the ball will continue to bounce infinitely often, find the total distance the ball will travel.

a) $9/2\ m$
b) $18\ m$
c) $9\ m$
d) $12\ m$
e) ∞

24) If $xy' + 3y = 4x^2$ and $y(1) = 2$ then $y =$

a) $\frac{1}{3}(4x^2 - x)$
b) $\frac{4x^2}{5} + \frac{6}{5x^3}$
c) $\frac{4x^2}{5} + \frac{6}{5}$
d) $\frac{1}{2}x^2$
e) $\frac{4x^2}{5} + \frac{x^3}{5}$

25) $\int \frac{dx}{x^3+x} =$

a) $\ln\sqrt{x^4+x^2}+C$

b) $\frac{1}{3x^2+1}\ln(x^3+x)+C$

c) $\ln\sqrt{\frac{x^2}{x^2+1}}+C$

d) $\ln|x|+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C$

e) $\ln|x|+\tan^{-1}x+C$

26) If $f(x)=\sin(\cos(\tan x))$ then $f'(x)=$

a) $\cos^2(\tan x)\sec^2 x$

b) $-\sin x(\sin(\tan x)\sec^2 x+\cos(\tan x)(\cos x)$

c) $-\cos(\sin(\sec^2 x))$

d) $\sin x\cos x\sec^2 x-\sin^2 x\tan x+\cos^2 x\tan x$

e) $-\cos(\cos(\tan x))\sin(\tan x)\sec^2 x$

27) An integer $n>1$ has the property that n divides evenly into $35m+26$ and $7m+3$ for some integer m. What is n?

a) 2

b) 3

c) 5

d) 8

e) 11

28) Two lines through the origin that are tangent to the curve $x^2-4x+y^2+3=0$ are

a) $y=x$ and $y=-x$

b) $y=2x$ and $y=-2x$

c) $y=\frac{\sqrt{3}}{3}x$ and $y=-\frac{\sqrt{3}}{3}x$

d) $y=\sqrt{2}x$ and $y=-\sqrt{2}x$

e) $y=\frac{x}{3}$ and $y=-\frac{x}{3}$

29) Find $\frac{d^2y}{dx^2}$ if $2xy-y^2=3$.

a) $-\frac{y}{x-y}$

b) $\frac{3xy-x^2-y^2}{(x-y)^3}$

c) $\frac{2xy-y^2}{(x-y)^3}$

d) $\frac{2y-x}{(x-y)^2}$

e) $\frac{y}{x}$

30) Big Ben is the nickname for the clock tower of the Palace of Westminster in London. The minute hand of the clock is 4.3 meters long. Starting from the moment the hand is pointing straight up, approximately how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

a) $55.47\ m^2/s$
b) $12.9\ m^2/s$
c) $3.04\ m^2/s$
d) $0.97\ m^2/s$
e) $0.23\ m^2/s$

31) Find the sum of the infinite series: $2 + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots$

a) 6
b) e
c) e^2
d) $e^2 - 1$
e) $94/15$

32) $\lim_{x\to\infty}\left(\sqrt{x^2 - 5x} - x\right) =$

a) $-\infty$
b) 0
c) -5
d) $-5/2$
e) $+\infty$

33) If the system of equations below has a unique solution, what can be said about the coefficients c and d?

$$\begin{aligned} 2x_1 + 4x_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned}$$

a) $c = 2d$
b) $c \neq 2d$
c) $c \neq d$
d) $d = 2c$
e) $d \neq 2c$

34) An object moves along the coordinate line with position at time t is given by the function $s(t) = 2t^3 + 3t^2 - 12t + 9$. On which time interval(s) is the object speeding up?

a) $[0, 1)$ only
b) $[0, \infty)$ only
c) $(1, \infty)$ only
d) $(1/2,1)$ and $(2, \infty)$
e) $[0, 1/2)$ and $(1, \infty)$

35) A credit card company requires that your update your password for using your card at an ATM machine. Your password must be a 4-digit number (using any digit from 0 to 9), and cannot consist of four consecutive numbers (like 3456 or 8765) and cannot consist of the same value repeated four times (like 5555). Given these restrictions, how many different passwords are possible?

a) 10,000
b) 9,985
c) 9,976
d) 6,480
e) 3,360

36) A vector orthogonal to both $\mathbf{u} = \langle 1, -4, 7 \rangle$ and $\mathbf{v} = \langle 2, 2, 0 \rangle$ is

a) $\langle -14, 14, 10 \rangle$
b) $\langle 2, -8, 0 \rangle$
c) $\langle 3, -2, 7 \rangle$
d) $\langle -14, 14, -6 \rangle$
e) $\langle -2, 8, 0 \rangle$

37) $\int_1^2 \frac{dx}{1-x} =$

a) $-\ln 2$
b) 0
c) $-\infty$
d) $+\infty$
e) $\ln 2$

38) Hooke's law states that under appropriate conditions, a spring that is stretched x units beyone its natural length pulls back with a force kx, where k is a constant that depends on factors such as the material and thickness of the spring. If a spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length, how far beyond its natural length can the spring be stretched with 25 J of work?

a) $2\ m$
b) $5\ m$
c) $0.5\ m$
d) $10\ m$
e) $12.5\ m$

39) Solve for x: $\frac{e^x - e^{-x}}{2} = 3$

a) $x = 6$

b) $x = \ln(3 + \sqrt{10})$

c) $x = \ln(3 + 2\sqrt{2})$

d) $x = \ln 6$

e) No solution exists in the real numbers

40) Most airlines require that suitcases used as carry-on luggage be such that the sum of the three dimensions (length, width, and height) is at most 115 centimeters. Suppose a rectangular suitcase is twice as long as it is wide. What is the length of such a suitcase if it meets the airline carry-on requirement and has the greatest possible volume?

a) 38.33 cm

b) 25.56 cm

c) 89.44 cm

d) 28.75 cm

e) 51.1 cm