THIRTY-FIRST ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS 1987

Scoring formula: 4R - W + 40

Prepared by:
Mathematics Department
Tennessee Technological University
Cookeville, TN
William Jones, Co-ordinator

Edited by: Larry Bouldin, Roane State Community College

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

Contributors to TMTA for Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State Community College, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
Sears, Madison, Tennessee
TRW, Ross Gear Division, Lebanon, Tennessee

1. Let x and y represent any real numbers for which $f(x,y) = \sqrt{25x^3y^2}$ is defined. Which of the following is not equal to f(x,y)?

   a) $5xy\sqrt{x}$

   b) $5x\sqrt{xy^2}$

   c) $5x^{3/2}|y|$

   d) $5(x^3y^2)^{1/2}$

   e) All are equal to f(x,y).

2. How many of the following curves are tangent to $x^2 - y^2 = 16$?

   1) $x=4$
   2) $y=x$
   3) $x=y^2+4$
   4) $x^2+y^2=16$
   5) $(x-5)^2 + y^2=1$

   a) 1
   b) 2
   c) 3
   d) 4
   e) 5

3. The real solution set of the equation $3^{2x}-3^x-6=0$ is

   a) $\{3,-2\}$

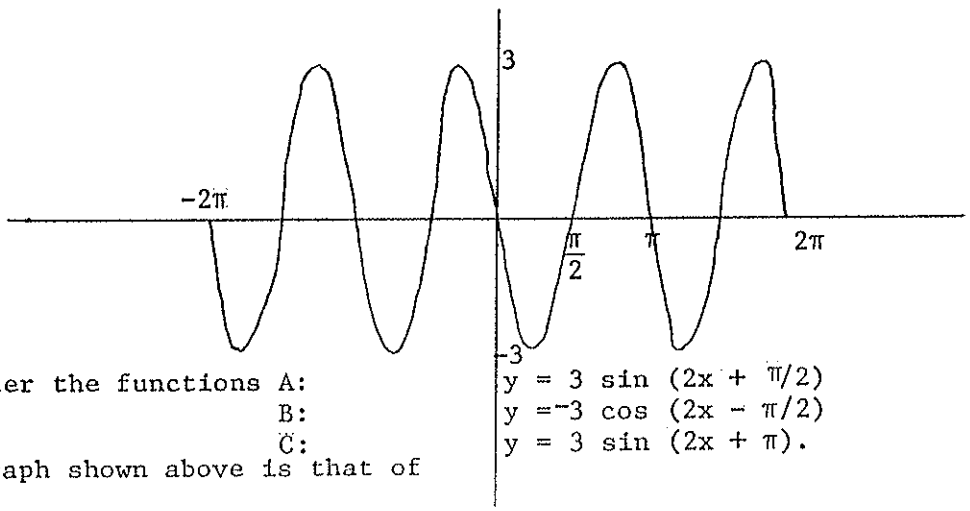
   b) $\{1\}$

   c) $\{1, \log_3(-2)\}$

   d) $\{1, \frac{1}{\log_3 2}\}$

   e) $\{1, \log_3 1/2\}$

4.



Consider the functions A: $y = 3 \sin (2x + \pi/2)$
B: $y = -3 \cos (2x - \pi/2)$
C: $y = 3 \sin (2x + \pi)$.

The graph shown above is that of

a) A only, b) B only, c) C only, d) A and B, e) B and C

5. The point whose polar coordinates are $(5, -\frac{\pi}{6})$ is the same as the point whose polar coordinates are

   a) $(-5, \frac{\pi}{6})$

   b) $(-5, \frac{5\pi}{6})$

   c) $(5, \frac{-5\pi}{6})$

   d) $(-5, \frac{-\pi}{6})$

   e) $(5, \frac{5\pi}{6})$.

6. The sum of three integers which form an increasing geometric progression is 65, and the sum of their reciprocals is 13/45. Find the common ratio.

   a) 5

   b) 1/2

   c) 1/3

   d) 3

   e) 2

7. $\lim_{x \to 2} \frac{|x-2|}{x-2}$

   a) =1

   b) =-1

   c) =0

   d) = 1/2

   e) does not exist.

8. The graph of $y = 4x^2 + 4\sqrt{2}kx + k + 3$ will intersect the x-axis in two distinct points for which of the following values of k?

   a) $k > -3$

   b) $-1 < k < \frac{3}{2}$

   c) $k < -1$ or $k > \frac{3}{2}$

   d) $k < -\frac{3}{2}$ or $k > 1$

   e) no real values of k.

9. Which of the following is not an identity for $\cos 2\theta$?

a) $\cos 2\theta = 2 \cos \theta \cos 3\theta - \cos 4\theta$

b) $\cos 2\theta = (\sin 2\theta)(\cot \theta) - 1$

c) $\cos 2\theta = 1 - 2 \sin \theta \cos \theta \tan \theta$

d) $\cos 2\theta = \dfrac{2}{\tan\left(\frac{\pi}{4} - \theta\right) + \cot\left(\frac{\pi}{4} - \theta\right)}$

e) $\cos 2\theta = \sqrt{\dfrac{1 - \cos 4\theta}{2}}$

10. $\int \ln x \, dx =$

a) $1/x + C$

b) $\dfrac{(\ln x)^2}{2} + C$

c) $x(\ln x) - x + C$

d) $x + \ln x + C$

e) $-x(\ln x) + x + C$

11. A line segment is drawn connecting the points (4,2) and (9,3) on the graph of $y = \sqrt{x}$. If a line parallel to this line segment is tangent to the curve, the point of tangency is

a) $\left(\frac{25}{4}, \frac{5}{2}\right)$

b) $\left(\sqrt{\frac{5}{2}}, \sqrt[4]{\frac{5}{2}}\right)$

c) $\left(\frac{4}{25}, \frac{2}{5}\right)$

d) $\left(\frac{49}{9}, \frac{7}{3}\right)$

e) $\left(\frac{1}{100}, \frac{1}{10}\right)$.

12. The fifth term in the expansion of $(a + bi)^7$ is

a) $35a^3b^4$

b) $-35a^3b^4$

c) $21a^2b^5i$

d) $-21a^2b^5i$

e) $35a^3b^4i$

13. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}$ is the power series representation for

a) $\ln x$

b) $\sin x$

c) $\cos x$

d) e^x

e) $\dfrac{1}{1 + x}$

14. If Joe can beat Tom by one-fifth of a mile in a three mile race, and Tom can beat Sam by one-tenth of a mile in a three mile race, by what distance can Joe beat Sam in a three mile race (to 2 decimal places)?

a) .10 mile
b) .15 mile
c) .29 mile
d) .31 mile
e) .34 mile

15. The altitude of a regular tetrahedron whose edge is e is

a) $\frac{e\sqrt{6}}{2}$
b) $\frac{e\sqrt{6}}{3}$
c) $\frac{e\sqrt{3}}{3}$
d) $\frac{2e}{3}$
e) $\frac{e\sqrt{3}}{2}$

16. The area of the bounded region under $y = x^2$ and above $y = x^3 - 6x$ is

a) $\frac{16}{3}$
b) $-\frac{16}{3}$
c) $\frac{63}{4}$
d) $\frac{253}{12}$
e) $\frac{125}{12}$

17. If $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{10 - x}$ are real-valued functions, then the domain of $f \circ g$ is the interval

a) $[6,+\infty)$
b) $(6,+\infty)$
c) $(-\infty,10]$
d) $[6,10]$
e) $[-2,2]$

18. Assuming that $y^4 + 3y - 4x^3 = 5x + 1$ defines, implicitly, a differentiable function f such that $y = f(x)$, find its derivative, y'.

a) $y' = \frac{12x^2 + 5}{4y^3 + 3}$
b) $y' = \frac{2 + 12x^2}{4y^3}$
c) $y' = \frac{5 + 12x^2 - 4y^3}{3}$
d) $y' = \frac{5 + 12x^2}{3y^2}$
e) $y' = \frac{12x^2}{y^3 + 1}$

19. Which of the following cannot be a zero of any polynomial p(x) with integer coefficients for which p(1) = 12?

a) 2
b) 3
c) 4
d) 5
e) 6

20. $[2(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12})]^4 =$

a) $8 + 8i\sqrt{3}$
b) $8 - 8i\sqrt{3}$
c) $8\sqrt{3} + 8i$
d) $\sqrt{3} - i$
e) $1 - i\sqrt{3}$

21. $\lim_{t \to 0} \frac{\tan t}{2t} =$

a) 0
b) 1/2
c) does not exist
d) 2
e) 1

22. Find $\int_{-1}^{2} x^{-3}\, dx$.

a) $\frac{3}{8}$
b) $\frac{-5}{8}$
c) $\frac{15}{64}$
d) ln 8
e) The integral is undefined.

23. A quadratic equation the roots of which are one half the roots of $x^2 - 6x + 7 = 0$ is

a) $2x^2 - 12x + 14 = 0$
b) $4x^2 - 12x + 1 = 0$
c) $4x^2 - 12x + 7 = 0$
d) $x^2 - 12x + 28 = 0$
e) $4x^2 - 12x - 7 = 0$

24. If a right circular cylinder is circumscribed about a sphere, the surface area of the sphere, compared with the total surface area of the cylinder (including top and bottom) is

a) one-third as great
b) one-half as great
c) two-thirds as great
d) the same
e) twice as great.

25. The volume in cubic units of the solid formed when the region bounded by the x-axis, $x = 1$, and $y = x^3$ is rotated about the x-axis is:

a) π d) $\pi/7$

b) $\pi/2$ e) $\pi/4$

c) $\pi/6$

26. For what value or values of k (if any) will there be infinitely many solutions to

$$\begin{cases} x + 3y - z = 2 \\ 5x + (k+1)y - 3z = 3 \\ 7x - 13y + kz = 0? \end{cases}$$

a) No value of k d) $k = -3$ and $k = 10$

b) $k = -3$ e) $-3 \leq k \leq 10$.

c) $k = 10$

27. If $\tan \theta = x$, then $\sin 2\theta =$

a) $\frac{2x}{1+x^2}$ d) $\frac{2x}{1-x^2}$

b) $\frac{x}{\sqrt{1+x^2}}$ e) $\frac{1}{\sqrt{1+x^2}}$

c) $\frac{1-x^2}{1+x^2}$

28. A field in the shape of a sector of a circle is to be fenced with 1200 feet of fencing. The central angle θ (in radians) of the sector which will give the maximum possible area is:

a) $\frac{\pi}{4}$ d) 2

b) $\frac{\pi}{3}$ e) $\frac{2\pi}{3}$

c) $\frac{\pi}{2}$

29. Find all the values of x in the interval $[0,2\pi]$ for which $2\cos^2 2x - 1 = 0$.

a) $\{\frac{\pi}{4}, \frac{5}{4}\pi\}$

b) $\{\frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi\}$

c) $\{\frac{\pi}{8}, \frac{3}{8}\pi, \frac{5}{8}\pi, \frac{7}{8}\pi, \frac{9}{8}\pi, \frac{11}{8}\pi, \frac{13}{8}\pi, \frac{15}{8}\pi\}$

d) $\{\frac{\pi}{2}, \frac{3}{2}\pi\}$

e) $\{0, \pi\}$

30. $\dfrac{x+4}{(x-4)^{1/3}+2} =$

a) $(x-4)^{2/3} - 2(x-4)^{1/3} + 4$

b) $(x-4)^{2/3} + 2(x-4)^{1/3} + 4$

c) $(x-4)^{2/3} + 4$

d) $(x-4)^{2/3} - 4$

e) $(x+4)^{2/3}$

31. If $\displaystyle\int_1^x \frac{dt}{\sqrt{4-t^2}} = \frac{\pi}{4}$, the value of x is

a) $\dfrac{\sqrt{6}+\sqrt{2}}{2}$

b) $\sqrt{3}$

c) $\sqrt{4 - (\sqrt{3}+\frac{\pi}{8})^2}$

d) $\pm\sqrt{4 - (\sqrt{3}+\frac{\pi}{8})^2}$

e) $\pm\sqrt{4 - e^{\pi/2} + 2\ln 3}$

32. If $\dfrac{5x^2 - 5x - 4}{(x-3)(x^2+4)} \equiv \dfrac{A}{x-3} + \dfrac{Bx+C}{X^2+4}$, then B =

a) 2

b) -2

c) 3

d) -3

e) 4

33. The region bounded by the graph of $y=2x - x^2$ and the x-axis is revolved about the y-axis. The volume in cubic units of the resulting solid is:

a) $5\pi/3$

b) $7\pi/16$

c) $8\pi/3$

d) $3\pi/4$

e) $\pi/9$

34. The sum of all of the numbers between 10 and 200 that are exactly divisible by 7 is

a) 1330

b) 2835

c) 406

d) 1470

e) 2842.

35. A ball rebounds to 2/3 of its initial height when dropped. If the ball is dropped from a height of 3 feet, what is the total distance it travels before coming to rest?

a) 6 feet

b) 9 feet

d) 18 feet

e) 25 feet

36. If matrix $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and the -1 power denotes the multiplicative inverse of the matrix, then which of the following is true?

a) $(A + B)^{-1} = A^{-1} + B^{-1}$

b) $A \times B = B \times A$

c) $A^{-1} \times B^{-1} = (A \times B)^{-1}$

d) $A \times B^{-1} = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}$

e) $A^{-1} \times B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

37. The function f is continuous for all x and has a local (relative) maximum at 0 and a local (relative) minimum at 1. Label statements I, II, and III as always true, sometimes true, or never true.

I) $f'(0) = 0$

II) f has an inflection point between 0 and 1

III) $f'(1) > 0$.

a) I. Always true II. Always true III. Never true

b) I. Sometimes true II. Always true III. Never true

c) I. Sometimes true II. Always true III. Sometimes true

d) I. Sometimes true II. Sometimes true III. Never true

e) I. Sometimes true II. Sometimes true III. Sometimes true

38. A formula which describes the growth of a bacterial colony is given by $N = N_o\, a^{rt}$, where N is the number of bacteria at the end of time t, N_o is the original number of bacteria, r is the rate of growth, and a is constant. If $\log N$ is the logarithm of N to the base 10, solving for t yields

a) $\dfrac{\log(N-N_o)}{r \log a}$

b) $\dfrac{\log(N-N_o)}{(\log a)^r}$

c) $\dfrac{\log N+\log N_o}{(\log a)r}$

d) $\dfrac{\log(N-N_o)}{\log(a^r)}$

e) $\dfrac{\log N - \log N_o}{r \log a}$

39. An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 inches wide and 21 inches long by cutting out a square from each corner and then bending up the sides. What is the dimension of the corner square that will produce a box having the largest possible volume?

a) 28/3
b) 2
c) 1
d) 4.5
e) 3

40. $\lim_{x \to \infty} \sqrt{x^2 + 5x} - x =$

a) 0
b) 5/2
c) ∞
d) 1
e) 5