THIRTIETH ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS 1986

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Scoring formula: 4R - W + 40

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

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TRW, Ross Gear Division, Lebanon, Tennessee

TMTA MATHEMATICS CONTEST FOR HIGH SCHOOL STUDENTS
ADVANCED TOPICS TEST
1986

1. $\sqrt{4a^2 + 4b^2} =$

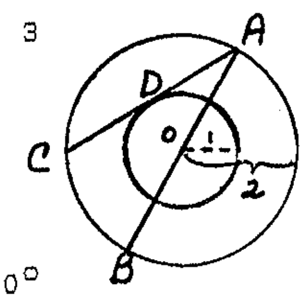
(A) $2ab$ (B) $2a + 2b$ (C) $8\sqrt{a^2 + b^2}$

(D) $4\sqrt{a^2 + b^2}$ (E) $2\sqrt{a^2 + b^2}$

2. Consider two concentric circles with center O, one of radius 1 unit and the other of radius 2 units. Let A and B be the endpoints of a diameter of the larger circle. Let C be another point on the larger circle such that the chord AC is tangent to the smaller circle at point D. What is the ratio of the area of ΔABC to the area of ΔAOD?

(A) 4 (B) 2 (C) 3

(D) $2\sqrt{3}$ (E) $\frac{2}{\sqrt{3}}$



3. Evaluate the following: $\tan^2 120^\circ + 3\cos^2 210^\circ - \sin^2 300^\circ$

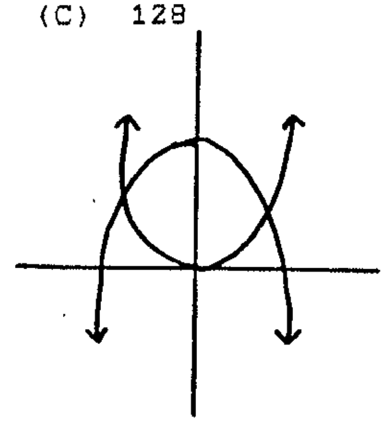
(A) 3 (B) $3 + \sqrt{3}$ (C) 4.5

(D) 6 (E) $\frac{3}{2}$

4. Find the area enclosed by $y = 3x^2$ and $y = 64 - x^2$.

(A) $\frac{1024}{3}$ (B) 64 (C) 128

(D) $\frac{704}{3}$ (E) $\frac{512}{3}$



5. Find $\frac{dy}{dx}$ for $y = \frac{(x + 1)^3}{x^3 + 1}$

(A) $\frac{(x + 1)^2}{x^2}$

(B) $\frac{3(x + 1)^2(1 - x^2)}{x^3 + 1}$

(C) $\frac{3(x + 1)^2(1 - x^2)}{(x^3 + 1)^2}$

(D) $\frac{3(x + 1)^2}{(x^3 + 1)^2}$

(E) $\frac{(x + 1)^4}{x^4 + x}$

6. A large box contains a total of 16 parts, of which 10 are good and 6 are defective. An inspector randomly draws a sample of 3 parts without replacement from the box. What is the probability that at least 2 of the 3 parts will be defective?

(A) $\frac{15}{56}$

(B) $\frac{17}{56}$

(C) $\frac{7}{1430}$

(D) $\frac{1}{8}$

(E) $\frac{1}{28}$

7. Find $\lim_{x \to 4} \frac{|x - 4|}{x - 4}$

(A) 1

(B) -1

(C) 0

(D) 4

(E) Limit does not exist

8. An orange farm yields an average of 128 bushels of oranges per tree when 20 trees are planted on an acre of ground. Each time one more tree is planted per acre, the yield decreases by 4 bushels per tree. How many trees should be planted to give the highest yield per acre.

(A) 25

(B) 24

(C) 22

(D) 28

(E) 26

9. If $abcdef \neq 0$, then $\lim_{x \to 0} \left(\lim_{y \to 0} \dfrac{ax^2 + bxy + cy^2}{dx^2 + exy + fy^2} \right)$ would be

(A) 0 (B) $\frac{a}{d}$ (C) $\frac{c}{f}$

(D) $\frac{ac}{df}$ (E) The limit does not exist

10. If $y = \sin(\tan x)$ for $0 < x < \frac{\pi}{2}$ then y' in simplified form is

(A) $\sin(\sec^2 x) + \cos(\tan x)$ (B) $\cos(\sec^2 x)$

(C) $\sec x$ (D) $\sec^2 x \cos(\tan x)$

(E) $\sec^2 x \sin x$

11. If $x = t^3$ and $y = t^2$ for any real number t, $\int_0^8 xy\,dx$ would be

(A) $\frac{8^6}{6}$ (B) $\frac{2^6}{6}$ (C) $3(8^7)$

(D) $3(2^5)$ (E) 8

12. Consider a right circular cone of height 10 units. A plane parallel to the base at a level 6 units from the base of the cone intersects the cone in a circle of area 6π. What is the volume of the cone?

(A) 375π (B) 150π (C) 125π

(D) 135π (E) 45π

13. The tangents to the curve of

$$f(x) = \frac{1}{3}x^3 - 3x^2 + 5x + 10$$

are horizontal when x has the values

(A) 2, 4 (B) 3, 5 (C) 4

(D) 1,4 (E) 1,5

14. For a certain group of students enrolled in a college, course A and course B are electives. The probabilities that a student, randomly selected from the group, will take course A, course B, and both course A and course B are respectively, 5/6, 3/4, and 2/5. Find the probability that a student who takes course A will also take course B.

(A) 5/8 (B) 12/25 (C) 8/15

(D) 9/40 (E) 3/10

15. At any point (x, y) on the circle $x^2 + y^2 = 16$, the value of $\frac{d^2y}{dx^2}$ could be calculated by

(A) $\frac{16}{(16 - x^2)^{3/2}}$ (B) $-\frac{16}{(16 - x^2)^{3/2}}$

(C) $\frac{16}{y^3}$ (D) $-\frac{16}{y^3}$

(E) $\frac{32}{(16 - x^2)^{3/2}}$

16 Let m and m be two distinct parallel lines. Let r and s be two intersecting lines which lie in the same plane as m and n. Suppose the point of intersection of lines r and s does not lie on either line m or line n. Let Triangle #1 be the triangle with vertices $m \cap r$, $m \cap s$, and $r \cap s$. Let Triangle #2 be the triangle with vertices $n \cap r$, $n \cap s$, and $r \cap s$. If Triangle #1 is isosceles, complete the following statements with always, sometimes, or never.

Triangle #2 is ___I___ isosceles.

Triangle #2 is ___II___ congruent to Triangle #1.

Triangle #2 is ___III___ similar to Triangle #1.

(A) I.sometimes, II.never, III.sometimes

(B) I.always, II.sometimes, III.sometimes

(C) I.always, II.never, III.always

(D) I.sometimes, II.never, III.always

(E) I.always, II.never, III.sometimes

17. The planar region bounded by $y = 2 - x^2$ and $y = 1$ is revolved around the x-axis. The resulting volume is

(A) $\frac{8\pi}{15}$ (B) $\frac{16\pi}{15}$ (C) $\frac{28\pi}{15}$

(D) $\frac{56\pi}{15}$ (E) $\frac{28\pi}{15}$

18. The legs of a right triangle are measured to be 8 and 15 with a possible error of 3% . Approximate the maximum error in stating the area as 60.

(A) 0.6 (B) 1.2 (C) 1.8

(D) 3.6 (E) 6.0

19. There are twice as many cats as there are dogs, and twice as many mice as there are cats. If the number of dogs and mice together is 750, then the number of cats is

(A) 150 (B) 300 (C) 600

(D) 75 (E) 400

20. Find $f'(x)$ for $f(x) = (\sin x) \ln(x^r)$

(A) $(x \cos x) \ln x + \sin x$ (B) $\frac{r \sin x}{x}$

(C) $(\cos x) \ln x^r$ (D) $\frac{r}{x} [(x \cos x) \ln x + \sin x]$

(E) $\cos x \frac{r}{x}$

21. The water in a conical tank, with vertex downward, and sides inclined at 45°, is rising at the rate of 2 ft./sec. At what rate is the volume of the water increasing when the water level is 4 ft. from the vertex?

(A) 32 cu.ft./sec. (B) 16π cu.ft./sec (C) 96 cu.ft./sec.

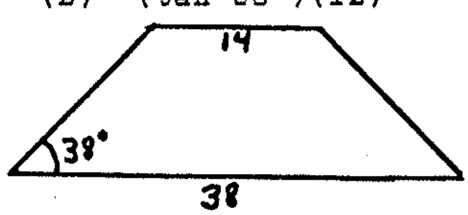
(D) 32π cu.ft./sec. (E) 12π cu.ft./sec.

22. An isosceles trapezoid has bases of 38 m and 14 m. Each base angle has a measure of 38°. Find the length of each of the equal sides expressed in terms of a trigonometric function.

(A) $(\sec 38^\circ)(12)$ (B) $(12)/(\sin 38^\circ)$

(C) $(\cos 38^\circ)(12)$ (D) $(\tan 38^\circ)(12)$

(E) $(24)/(\cos 38^\circ)$



23. Evaluate $\displaystyle\int_{e}^{e^2} \frac{1}{x \ln x} dx$

(A) $\dfrac{1}{2e^2 \ln e}$ (B) $\dfrac{1}{e \ln e}$

(C) $\ln 2$ (D) $2 \ln e$ (E) $\ln e^2 - \ln e$

24. If $2 = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$ then x is

(A) 8 (B) 4 (C) 2

(D) $\frac{1}{2}$ (E) undefined

25. $\displaystyle\lim_{x \to 0^+} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}$ is equal to

(A) $a + b$ (B) $\frac{1}{2}(a + b)$ (C) $\ln (ab)$

(D) ab (E) $\sqrt{ab}$

26. If $f(x) = \dfrac{|x^2 - x - 6|}{x + 2}$, which of the following is true?

(A) $f(2)$ is undefined (B) $\displaystyle\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

(C) $f(2) \neq \displaystyle\lim_{x \to 2} f(x)$ (D) $f'(2)$ does not exist

(E) $f(x)$ is continuous at $x = 2$

27. Evaluate $\int_0^1 \frac{dx}{4 - x^2}$

(A) $\frac{1}{6}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{2}$

(D) $\frac{\pi}{3}$ (E) π

28. Consider the matrix $A = \begin{bmatrix} x - 4 & -x & 2 \\ -5 & x - 2 & 1 \\ 2x & -3 & 6 \end{bmatrix}$. For what value of x does A <u>not</u> have an inverse?

(A) $\frac{6}{5}$ (B) 0 (C) $\frac{5}{7}$

(D) 1 (E) $\frac{7}{6}$

29. Evaluate $\int_0^2 \frac{x^3 + 8}{x + 2}\, dx$

(A) 6 (B) $\frac{20}{3}$ (C) $\frac{32}{3}$

(D) 10 (E) 4

30. If $f(x) = x^2 - 2$, then $f(f(f(1))) =$

(A) 0 (B) $3\sqrt{3}$ (C) 1

(D) 2 (E) -1

31. Find the indefinite integral $\int \frac{\cos x \, dx}{a^2 + \sin^2 x} dx$

(A) $\tan^{-1}\left(\frac{\sin x}{a}\right)$

(B) $\tan^{-1}(x) + c$

(C) $\ln(a^2 + \sin^2 x) + c$

(D) $\frac{1}{a} \tan^{-1}\left(\frac{\sin x}{a}\right) + c$

(E) $\tan(\sin x) + c$

32. The angle of depression from the top of a cliff 30 m high to a boat on a lake below the cliff is 30°. Find the distance of the boat from the base of the cliff.

(A) 30 m

(B) 60 m

(C) $30\sqrt{3}$ m

(D) 15 m

(E) $20\sqrt{3}$ m

33. In how many different ways can the manager of a baseball team arrange the batting order of the first four batters if he selects the four from the nine players in the starting lineup?

(A) 24

(B) 126

(C) 256

(D) 3024

(E) 6561

34. Find the absolute maximum and minimum values of

$$f(x) = x^3 - 6x^2 + 9x + 8$$

over the closed interval $[2, 4]$.

(A) 8, -8

(B) 12, 8

(C) 10, 8

(D) 12, -8

(E) 12, 10

35. Find all x such that $0 \leq x \leq 2\pi$ and $\sin x = -1/2$.

(A) $\frac{\pi}{6}, \frac{5\pi}{6}$ (B) $\frac{5\pi}{6}, \frac{7\pi}{6}$

(C) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (D) $\frac{-\pi}{6}$ (E) $\frac{7\pi}{6}$

36. Let A denote a matrix and A_{ij} denote the ij^{th} cofactor of A, where i denotes row and j denotes column. If i=2 and j=3, find A_{ij} for

$$A = \begin{bmatrix} -4 & 5 & 1 \\ 0 & -8 & -2 \\ 3 & -7 & 6 \end{bmatrix}$$

(A) $\begin{bmatrix} -4 & 5 \\ 3 & -7 \end{bmatrix}$ (B) 26 (C) -26

(D) 13 (E) -13

37. For what value(s) of the constant c is the graph of $x^2 - 2x - 2y^2 - 2y + 4z^2 - 8z = c$ a hyperboloid of two sheets?

(A) Only $c = 0$ (B) $c = \frac{-9}{2}$ (C) $c < \frac{-9}{2}$

(D) $c > \frac{-9}{2}$ (E) Only $c = 1$

38. $\sqrt{(A + B)^2 - 4AB} =$

(A) $A + B - 2\sqrt{AB}$ (B) $|A - B|$ (C) $A - 2B$

(D) $A - B$ (E) $\sqrt{A^2 - 4AB + B^2}$

39. The period for $y = 3 - 2\sin(\pi x - 2\pi)$ would be

(A) 2 (B) −2 (C) π

(D) 2π (E) 3

40. The surface area, including ends, of a right circular cylinder, with a fixed volume, is minimum when the relative dimensions are:

(A) height is equal to radius; $h = r$

(B) height is equal to twice radius; $h = 2r$

(C) height is equal to four times radius; $h = 4r$

(D) $h = r\sqrt[3]{\pi}$

(E) $h = r\sqrt[3]{4\pi}$