TWENTY-NINTH ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS 1985

Scoring formula: 4R - W + 40

Prepared by: Vanderbilt University
Mathematics Department
Nashville, Tennessee
Horace Williams, Coordinator

Edited by: Larry Bouldin, Roane State
Community College

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on the the next. The working time for the entire test is 80 minutes.

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1. Consider the transformation $K: P \leftrightarrow \frac{1}{2}P + P_0$ of a plane onto itself where $P = (x,y)$ and $P_0 = (3,-4)$. The image A' of the vector $A = (12,8)$ under K is:

   a) $A' = (6,4)$
   b) $A' = (9,0)$
   c) $A' = (7\frac{1}{2},2)$
   d) $A' = (13\frac{1}{2},6)$
   e) $A' = (15,4)$

2. A triangle has vertices (6,8), (2,-4) and (-3,8). The area is

   a) 40
   b) $40\sqrt{3}$
   c) 54
   d) $60\sqrt{2}$
   e) none of the above

3. The length of a tangent segment from the point (-8,3) to the circle whose equation is $x^2 + y^2 - 14x + 10y + 10 = 0$ is:

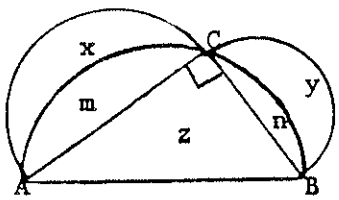
   a) $17\sqrt{2}$
   b) 17
   c) $15\sqrt{2}$
   d) 15
   e) $\sqrt{353}$

4. The coordinates of the centroid of a triangle with vertices (-4,0), (6,0) and (3,6) are:

   a) (3,4)
   b) (5/2,3)
   c) (2,2)
   d) (5/3,2)
   e) (1,3)

5. Semicircles are drawn on each side of right triangle ABC, with right angle at C as shown in the figure. x,y,z,m and n are the areas of the five disjoint regions as shown. The area z of the triangle equals

   a) x + y
   b) m + n
   c) x + m
   d) x + y + m + n
   e) y + n



6. The length of each chain supporting the seat of a child's swing is 6 feet. When the swing is at its highest points forward and backward, the angle so formed in $2\pi/3$ radians. How far does the child travel in one trip between these high points?

   a) $6\sqrt{2}$ feet
   b) $3\sqrt{3}$ feet
   c) 3 feet
   d) 6π feet
   e) 4π feet

7. On what portion of the interval $[0,2\pi]$ is it always true that $\sin\theta > \cos\theta$?

   a) $(0,\pi)$
   b) $(\pi/4,5\pi/4)$
   c) $(\pi/2,3\pi/2)$
   d) $(3\pi/4,7\pi/4)$
   d) $(-\pi/2,\pi/2)$

8. The period for the function $f(x) = \sin 3x + \cot \frac{3x}{4}$ is

   a) $2\pi/3$
   b) $4\pi/3$
   c) π
   d) 2π
   e) 3π

9. The complete solution set for the equation $2\cos^2 2x - 4\cos 2x = -\frac{3}{2}$ is

   a) $\{\pi/3 + n\pi\}$
   b) $\{\pm\pi/3 + 2n\pi\}$
   c) $\{\pi/6 + n\pi\}$
   d) $\{\pm\pi/6 + n\pi\}$
   e) $\{\pm\pi/6 + 2n\pi\}$

10. Evaluate: $\tan[2\cos^{-1}(\frac{1}{3})]$. [Note: $\cos^{-1}(\frac{1}{3}) = \text{arc cos } (\frac{1}{3})$].

    a) $2\sqrt{2}$
    b) $-2\sqrt{2}$
    c) $-\frac{4}{7}\sqrt{2}$
    d) $\frac{4}{9}\sqrt{2}$
    e) $4\sqrt{2}$

11. A root of $x^3 - x - 1 = 0$ lies between $x = 1$ and $x = 2$. An initial approximation to the root of 1.00 is used with Newton's method. The second approximation by this method is

    a) 2.00
    b) 1.00
    c) $1/2 + \sqrt{5}/2$
    d) 1.33
    e) 1.50

12. Approximate $\sqrt{10}$ to two decimal places using a differential approximation to the function $y = \sqrt{x}$ with $x_0 = 9$, $dx = 1$.

a) 3.20
b) 3.17
c) 3.16
d) 3.15
e) 3.10

13. The inverse of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$ is

a) $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$
b) $\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & -2/3 \end{bmatrix}$
c) -3
d) $\begin{bmatrix} 2/3 & 1/3 \\ -1/3 & -2/3 \end{bmatrix}$
e) does not exist

14. The equation $x^3 + x - 1$ has a real root between

a) -2 and -1
b) -1 and 0
c) 0 and 1
d) 1 and 2
e) has no real roots.

15. The matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & k \\ -2 & 3 & 1 \end{bmatrix}$ is symmetric if and only if

a) k is any real number
b) $k = 2$
c) $k = -\frac{13}{7}$
d) $k = 3$
e) none of the above

16. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & k \\ -2 & 3 & 1 \end{bmatrix}$ be the coefficient matrix for a system of linear equations. The system will <u>not</u> have a unique solution for

a) $k = 0$
b) $k = 2$
c) $k = 1$
d) $k = 3$
e) $k = -\frac{13}{7}$

17. Consolidated Industries has come under pressure to eliminate its discriminatory hiring practices. Company officials have agreed that during the next five years 60% of their new employees will be females and 30% will be black. One out of four new employees, though, will be white males. What percentage of black females is the company committed to hiring?

a) 18%
b) 10%
c) 30%
d) 20%
e) none of the above

18. An urn contains 15 chips, numbered 1 through 15. Four are chosen without repetition. What is the probability the second largest chip selected is a 9?

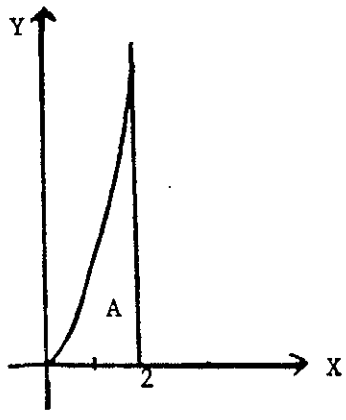
a) $168/\binom{15}{4}$
b) $\binom{9}{2}/\binom{15}{4}$
c) $588/\binom{15}{4}$
d) $9!/15!$
e) none of the above

19. In a forthcoming movie, Friday the 13th (VIII), Jason's great-grandson is a psychotic trying to dismember a group of 8 camp counselors, comprised of 4 men and 4 women. In how many orders can he dispatch all the men before any of the women?

a) 40,320
b) 256
c) 576
d) 20,160
e) 96

20. The volume (in cubic units) of the solid generated by rotating about the y-axis the region A bounded by $y = x^3$, $x = 2$, and $y = 0$ (see figure) is

a) $\frac{64\pi}{5}$
b) $\frac{128\pi}{7}$
c) $\frac{32\pi}{3}$
d) 8π
e) 16π



21. If $f(x) = \sin^5(x^2 + 1)$, then $f'(x)$ is

a) $\cos^5(x^2 + 1)$

b) $2x \cos^5(x^2 + 1)$

c) $10x \cos^4(x^2 + 1)$

d) $10x \sin^4(x^2 + 1) \cos^4(x^2 + 1)$

e) $10x \sin^4(x^2 + 1) \cos(x^2 + 1)$

22. The focus and directrix, respectively, of the parabola

$$y^2 - 4y - 12x - 8 = 0$$

are

a) $(-1,2)$ and $x = 2$

b) $(2,2)$ and $x = -4$

c) $(-1,2)$ and $y = 2$

d) $(-\frac{2}{3},0)$ and $y = 0$

e) $(-1,2)$ and $x = 0$

23. $\int (\ln x)^2 \, dx =$

a) $\frac{1}{x^2} + C$

b) $x(\ln x)^2 + C$

c) $x(\ln x)^2 - 2x \ln x + C$

d) $x(\ln x)^2 - 2x \ln x + 2x + C$

e) $\frac{(\ln x)^3}{3} + C$

24. Points A,B,C are fixed points on a circle with center O and radius 1. As shown in the figure, OC is a radius perpendicular to the diameter AB. Points X,Y are variable points, with Y on the arc between C and B and with XY perpendicular to AB at point X.

What is the length of XY when the area of triangle AXY is a maximum?

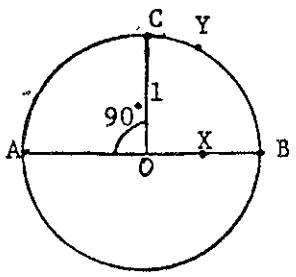
a) $\frac{1}{2}$

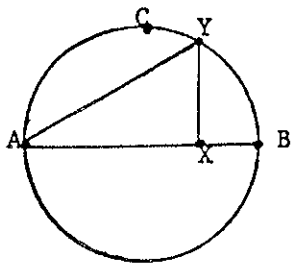
b) 1

c) $\frac{\sqrt{2}}{2}$

d) $\frac{\sqrt{3}}{2}$

e) $\sqrt{2}$





25. A box with an open top is to be made from a square piece of tin by cutting squares from each corner and turning up the edges. If the maximum volume is achieved when squares with edges of $3\frac{1}{3}$ inches are cut from each corner, then the edges of the original square piece of tin are

a) 10 inches
b) 15 inches
c) 20 inches
d) 25 inches
e) 30 inches

26. The points on the graph of $x^2 + y^2 = x + y + \frac{3}{4}$ at which the slope of the tangent line is -2 are

a) $(-\frac{1}{2},1)$ and $(-\frac{1}{2}, 0)$
b) $(\frac{3}{2},1)$ and $(\frac{3}{2},0)$
c) $(-\frac{1}{2},1)$ and $(\frac{3}{2},0)$
d) $(-\frac{1}{2},0)$ and $(\frac{3}{2},1)$
e) $(0,\frac{3}{2})$ and $(\frac{3}{2},1)$

27. Evaluate the definite integral

$$\int_1^2 \frac{x + 2}{3x^2}\, dx$$

a) $\frac{1}{3} \ln 2 + 1$
b) $6 \ln 4$
c) 11
d) $\frac{1}{3}(\ln 2 + 1)$
e) none of the above

28. $\int_1^e \frac{\ln x}{x}\, dx =$

a) $\frac{1}{e}$
b) 1
c) $\frac{1}{2}$
d) e
e) e^2

29. $\int_{-1}^2 |x^2 - 1|\, dx =$

a) 0
b) $\frac{1}{3}$
c) $\frac{14}{3}$
d) $\frac{8}{3}$
e) $\frac{19}{3}$

30. Determine m so that the region above the line $y = mx$ and below the parabola $y = 2x - x^2$ has an area of 36 sq. units.

a) $m = 2$
b) $m = 0$
c) $m = -2$
d) $m = -4$
e) none of the above

31. At midnight a truck travelling east at 80 ft/sec is in the middle of an intersection. At that moment a police car, travelling north towards the intersection at 60 ft/sec, is 240 ft. south of the intersection. The police car trains its spotlight on the truck. How fast is the spotlight revolving 2 seconds later?

a) $-\frac{3}{5}$ rad/sec
b) 0 rad/sec
c) $\frac{3}{5}$ rad/sec
d) $\frac{12}{25}$ rad/sec
e) 2 rad/sec

32. If $f(x) = \int_0^x \frac{1}{1 + t^2}\, dt$ then $f^{-1}(x) =$

a) tan x
b) arctan x
c) tanh x
d) sin x
e) arcsin x

33. The function $f(x) = \frac{x^2 - 4}{x + 2}$ is <u>not</u> continuous

a) at $x = 4$
b) at $x = 2$
c) at $x = -2$
d) at $x = 0$
e) anywhere

34. For the function $f(x) = \frac{x^2 - 4}{x + 2}$, $\lim_{x \to c} f(x)$

a) does not exist at $c = 4$
b) exists everywhere
c) does not exist at $c = 2$
d) does not exist at $c = -2$
e) does not exist at $c = 0$

35. The value of the function $f(x) = \frac{x^2 - 4}{x + 2}$ at $x = -2$

a) is -4

b) is 4

c) is 0

d) is 1

e) does not exist

36. Find: $\lim_{x \to 5^-} \frac{|x - 5|}{x - 5}$

a) 0

b) 1

c) -1

d) does not exist

e) none of the above

37. The derivative of $f(x) = \sin^2 x$ is

a) $\sin 2x$

b) $\cos^2 x$

c) $\tan^2 x$

d) $\sin x$

e) none of these

38. An oblique asymptote to the curve of $y = \frac{x^2 + 4}{x + 2}$ is

a) $x = -2$

b) $y = x - 2$

c) $y = 8$

d) $y = x + 2$

e) $x = \pm 2i$

39. For a function $f(x)$, if $f'(-2) = 8$ then

a) $f(x) = x^2 + 8$

b) $f(x) = \sin 2x$

c) $x = -2$ is a critical point of $f(x)$

d) the function is continuous at $x = -2$

e) the slope of a tangent to the curve at $x = -2$ is -4

40. Suppose a baseball pitcher has a 50% chance of throwing a strike on any given pitch. If the batter never swings, what is the probability he strikes out?

a) 21/32

b) 21/64

c) 1/2

d) 3/8

e) 2/3