TWENTY-THIRD ANNUAL MATHEMATICS CONTEST
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ADVANCED TOPICS TEST

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Scoring Formula: 4R - W + 40

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This test was prepared from a list of Advanced Topics questions submitted by Tennessee Technological University.

DIRECTIONS:

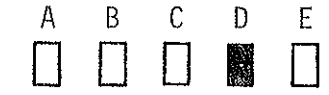
Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers; one and only one is correct. You are to work each problem, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. You must use a pencil with soft lead (No. 2 lead or softer). A sample problem follows:

1. If $2x = 3$, then x equals

   (a) 2/3 (b) 3 (c) 6

   (d) 3/2 (e) none of the above



The correct answer for the sample problem is 3/2, which is answer (d); so you would answer this problem by making a heavy black mark under space D as indicated above.

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do much wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

The answer sheets will be used for a statistical compilation and will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. The y-intercept of the line tangent to the curve with equation $y = x^2 - 4x + 5$ at the point on the curve with x-coordinate 3 is

   (a) 7

   (b) -4

   (c) -6

   (d) 8

   (e) 1

2. The value of $\tan^{-1}\sqrt{3}$ is

   (a) $\frac{\pi}{3}$

   (b) $\sqrt{3}$

   (c) $\frac{1}{\sqrt{3}}$

   (d) $\frac{\pi}{6}$

   (e) $\frac{7\pi}{6}$

3. The value of $\sin 15^0$ is

   (a) $\frac{1}{4}$

   (b) $\frac{\sqrt{2 + 2\sqrt{3}}}{2}$

   (c) $\frac{1}{2}$

   (d) $\frac{\sqrt{2 - \sqrt{3}}}{2}$

   (e) $\sqrt{2 - \sqrt{3}}$

4. The sum of the odd integers from 1 through 201 is

   (a) 10201

   (b) 10101

   (c) 10200

   (d) 10102

   (e) none of the above

5. Which of the following functions f satisfies the relation $f(x + y) = f(x) + f(y)$ for all real values of x and y?

   (a) $f(x) = x^2$

   (b) $f(x) = \sqrt{x}$

   (c) $f(x) = \sin x$

   (d) $f(x) = 3x + 2$

   (e) $f(x) = -x$

6. The number of diagonals of a polygon of 14 sides is

   (a) 42

   (b) 28

   (c) 154

   (d) 77

   (e) none of the above

7. The derivative $f'(x)$ of $f(x) = x|x|$ is equal to

   (a) $|x| + x$

   (b) $|x| - x$

   (c) $2x$

   (d) $2|x|$

   (e) none of the above

8. Consider the following pair of simultaneous equations in three unknowns. How many of the solutions to these equations have exactly one of the variables equal to zero?

$$2x + 3y - z = 6$$
$$x - y + z = 4$$

   (a) none

   (b) 3

   (c) 1

   (d) infinitely many

   (e) 2

9. A coin is biased so that a head is twice as likely to occur as a tail. If a coin is tossed three times, what is the probability of getting exactly two tails?

   (a) $\frac{2}{3}$

   (b) $\frac{2}{9}$

   (c) $\frac{1}{3}$

   (d) $\frac{4}{9}$

   (e) $\frac{8}{9}$

10. Five people are to be seated around a table. Two of the people, Bill and Shirley, must sit side by side. How many seating arrangements are possible?

   (a) 120

   (b) 24

   (c) 12

   (d) 48

   (e) none of the above

11. $\frac{1 - \tan x}{1 + \tan x}$ may be replaced by

   (a) $\frac{\sin 2x}{1 - \cos 2x}$

   (b) $\frac{\sin x - \cos x}{\cos x + \sin x}$

   (c) $\frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$

   (d) $\tan(x + \frac{\pi}{4})$

   (e) $\tan(\frac{\pi}{4} - x)$

12. The line whose equation is $y = mx + b$ is tangent to the circle whose equation is $x^2 + y^2 = 9$. The set of possible value(s) of b is

    (a) $\{3\}$

    (b) $\{3\sqrt{1 + m^2}\}$

    (c) $\{-3\sqrt{1 + m^2}\}$

    (d) $\{3\sqrt{1 + m^2}, -3\sqrt{1 + m^2}\}$

    (e) none of the above

13. The area of the triangular region with sides of length 56, 39, and 25 is

    (a) 420

    (b) 1092

    (c) 480

    (d) 700

    (e) none of the above

14. The absolute maximum of the function $f(x) = -x^3 + 3x - 8$ on the interval $-3 \leq x \leq 3$ occurs at the point where x =

    (a) -3

    (b) -1

    (c) 1

    (d) 3

    (e) none of the above

15. A bisection search is being conducted for a zero of the function $f(x) = 3x^2 + 2x - 10$ on the interval $-1 \leq x \leq 3$. The second approximation to a zero which the search determines is x =

    (a) 0

    (b) 1

    (c) 2

    (d) 3

    (e) none of the above

16. What value will be printed for Y by the following program?

```
10  READ  A, B, C
20  Y = (A ↑ 3 + 2 * B - C)/(A ↑ 2 - 2 * B + C/2)
30  PRINT Y
40  DATA 2, 3, 6
```

(a) 1

(b) 8

(c) 4

(d) -8

(e) none of the above

17. If n is an even positive integer, then $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^n$ =

(a) $\begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$

(b) $\begin{bmatrix} 2^{n/2} & 0 \\ 0 & 2^{n/2} \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2^{n/2} \\ 2^{n/2+1} & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & n \\ 2n & 0 \end{bmatrix}$

(e) none of the above

18. The determinant $\begin{vmatrix} 2 & -1 & 0 & 5 \\ 3 & 0 & 6 & -1 \\ 2 & 0 & 0 & -4 \\ -5 & 0 & 0 & 7 \end{vmatrix}$ is equal to

(a) -6

(b) 0

(c) -36

(d) 36

(e) none of the above

19. A boy has a penny, a nickel, a dime, and a quarter in his pocket. He takes two coins out, one after the other. What is the probability that he has more than 11¢ in his hand?

(a) $\frac{1}{2}$

(b) $\frac{3}{4}$

(c) $\frac{1}{4}$

(d) $\frac{2}{3}$

(e) none of the above

20. The graph of a quadratic function passes through the points (0, 1), (1, 0) and (4, 2). This quadratic function has a minimum value at the point where x =

(a) 0

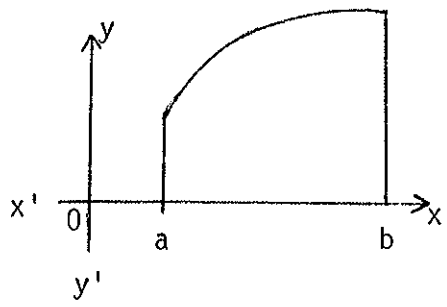
(b) 1.7

(c) 2.3

(d) -6.4

(e) none of the above

21. The graph of the function f on the interval $[a, b]$ appears in the following figure. Which of the following statements is true? (Assume that $f'(x)$ and $f''(x)$ exist for all $x \in (a, b)$)



(a) $f'(x) > 0$ and $f''(x) > 0$ for any $x \in (a, b)$

(b) $f'(x) > 0$ and $f''(x) < 0$ for any $x \in (a, b)$

(c) $f'(x) < 0$ and $f''(x) > 0$ for any $x \in (a, b)$

(d) $f'(x) < 0$ and $f''(x) < 0$ for any $x \in (a, b)$

(e) none of the above

22. Given a right triangle ABD with right angle at B, and $\overline{BC} \perp \overline{AD}$ at C. If CD = 2 and AC = 6, which of the following statements is false?

(a) $\Delta ABC \sim \Delta BDC$

(b) $BC = 2\sqrt{3}$

(c) $AB = 3\sqrt{3}$

(d) $\Delta CBD \sim \Delta BAD$

(e) $m(\angle CBD) = 30^0$

23. The values of a and b for which the line $y = ax + b$ is tangent to the curve $y = x^2$ at the point (1, 1) are

(a) $a = 2, b = -1$

(b) $a = -2, b = 1$

(c) $a = 0, b = 1$

(d) $a = 1, b = 0$

(e) none of the above

24. The total area of the faces of a regular hexahedron is equal to the total area of the faces of a regular octahedron. The ratio of the measure of an edge of the hexahedron to the measure of an edge of the octahedron is

(a) $\frac{1}{\sqrt[4]{3}}$

(b) $\frac{2\sqrt{3}}{3}$

(c) $\frac{\sqrt[4]{3}}{2}$

(d) $\frac{\sqrt{3}}{3}$

(e) $\frac{1}{\sqrt[3]{3}}$

25. The maximum value of the expression $2x + y$ for points in the first quadrant and bounded by the linear inequalities $3x + 2y \leq 6$ and $x + 2y \leq 4$ is

(a) 3.5

(b) 6

(c) 4

(d) 3

(e) none of the above

26. $\lim_{h \to 0} \frac{\frac{1}{x + h} - \frac{1}{x}}{h}$ is equal to

(a) 1

(b) 0

(c) undefined

(d) $\frac{1}{2x}$

(e) $-\frac{1}{x^2}$

27. The period of a pendulum is directly proportional to the square root of its length. A pendulum 9 feet long has a period of 4 seconds. If the pendulum is shortened to 4 feet its period will be

(a) 4 seconds.

(b) 9 seconds.

(c) $\frac{4}{3}$ seconds.

(d) $\frac{8}{3}$ seconds.

(e) none of the above.

28. If one root of the quadratic equation $ax^2 + bx + c = 0$ is 5, the other root is

(a) $-\frac{a}{b} + 5$

(b) $-\frac{a}{b} - 5$

(c) $-\frac{b}{a} - 5$

(d) $-\frac{b}{a} + 5$

(e) none of the above

29. If $f(x) = (x - 1)(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$, then $f'(1)$ is equal to

(a) 9

(b) $x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

(c) 0

(d) 10

(e) none of the above

30. A zero of the function $f(x) = x^3 - 2x + 8$ is being approximated by Newton's Method. If the last approximation was $x = 1$, then the next approximation will be $x =$

(a) 7

(b) 0

(c) 1

(d) -6

(e) none of the above

31. Let A, B, C denote 3 x 3 matrices with real numbers as entries. Which of the following properties always holds?

(a) $AB = BA$.

(b) If $AB = AC$ then $B = C$.

(c) $(AB)^2 = A^2B^2$.

(d) $(AB)C = A(BC)$.

(e) none of the above

32. How many times will step 50 in the following program be executed?

```
 5  T = 0
10  S = 0
15  I = 0
20  I = I + 1
30  S = S + I
40  IF I < 10 THEN GO TO 20
50  T = T + I
60  IF I < 12 THEN GO TO 20
70  PRINT S, T
```

(a) 10

(b) 12

(c) 3

(d) 2

(e) none of the above

33. The coefficient of x^9 in the expansion of $(x^2 - \frac{1}{x})^9$ is

(a) 9

(b) -9

(c) 84

(d) -84

(e) none of the above

34. If $x + 2$ is a factor of $2ax^3 + 3x^2 - ax + 2a$, then a equals

(a) 1

(b) 2

(c) 3

(d) 4

(e) none of these

35. In how many ways can 6 books be arranged on a shelf if 2 of them constitute a set and must always be kept together and in the same order?

(a) 720

(b) 120

(c) 240

(d) 24

(e) 6

36. A ray and an acute angle are coplanar. Which one of the following intersections is not possible?

(a) exactly one point

(b) exactly one ray

(c) exactly one segment

(d) the empty set

(e) exactly one line

37. In how many ways can change be made for a 48 cent purchase if the buyer has only a dollar bill and 3 pennies and the seller has only 6 dimes and 7 nickels?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

38. $\frac{\tan\theta + \cot\theta}{\sec\theta \sin\theta}$ may be replaced by

(a) $1 + \tan^2\theta$

(b) $\frac{1}{1 - \cos^2\theta}$

(c) $\frac{\sin^2\theta}{1 - \cos^2\theta}$

(d) $\frac{1 + \cos^2\theta}{1 - \cos^2\theta}$

(e) none of the above

39. Which of the following is not a complex 6th root of unity?

(a) -1

(b) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(c) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(d) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(e) 1

40. Line ℓ is perpendicular at the point with abscissa 3 to the line that contains the common chord of the circles whose equations are $x^2 + y^2 - 2x - y - 15 = 0$ and $x^2 + y^2 + 10x - 9y + 29 = 0$. The equation of ℓ is

(a) $3x - 2y - 28 = 0$.

(b) $2x - 3y + 16 = 0$.

(c) $2x + 3y - 36 = 0$.

(d) $3x + 2y - 32 = 0$.

(e) none of the above.