TWENTY-SECOND ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS TEST

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Scoring Formula: 4R - W

This test was prepared from a list of Advanced Topics questions submitted by the University of Tennessee at Knoxville.

DIRECTIONS:

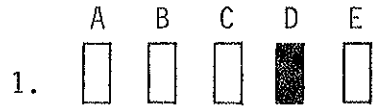
Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers; one and only one is correct. You are to work each problem, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. You must use a pencil with soft lead (No. 2 lead or softer). A sample problem follows:

1. If $2x = 3$, then x equals

   (a) 2/3. (b) 3. (c) 6.

   (d) 3/2. (e) none of these



The correct answer for the sample problem is 3/2, which is answer (d); so you would answer this problem by making a heavy black mark under space D as indicated above.

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers so it is advisable not to do much wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

The answer sheets will be used for a statistical compilation and will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 1 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. Two identical coins are randomly placed on a 4 x 4 checkerboard. The probability that both of the coins appear on corner squares (but not the same corner square) is:

   (a) $\frac{1}{120}$

   (b) $\frac{3}{64}$

   (c) $\frac{1}{20}$

   (d) $\frac{1}{16}$

   (e) $\frac{1}{8}$

2. Find the center and radius of the circle with equation $x^2 = 1 - y^2 - 4y$.

   (a) center: (0,2); radius: 5

   (b) center: (0, -2); radius: $\sqrt{5}$

   (c) center: (2, 0); radius: 5

   (d) center: (0, 0); radius: 2

   (e) center: (0, -2); radius 5

3. If $\cos (x + 45^0) = \sin (2x - 60^0)$ a possible value of x is.

   (a) 105^0

   (b) 35^0

   (c) 25^0

   (d) 65^0

   (e) 195^0

4. The graph of $y = \sin x + \cos (x + \frac{\pi}{2})$ would

   (a) be a straight line

   (b) be congruent to $y = \sin x + \cos x$

   (c) be the mirror image of $y = \tan x$

   (d) have amplitude $|\pm 1|$ and period $\frac{\pi}{2}$

   (e) None of the above

5. Among the values of A which satisfy the equation $1 + \sin A = \cos^2 A$ is

   (a) $\frac{\pi}{2}$

   (b) $\frac{3\pi}{2} + 2n\pi$

   (c) arcsin 1

   (d) $\frac{3\pi}{2} + \pi$

   (e) $-\frac{3\pi}{2}$

6. Three letters are selected one at a time, without replacement, from the letters of the word, MATHEMATICS. The probability that the letters in the order selected spell the word, HEM, is

   (a) $\frac{2}{1331}$

   (b) $\frac{1}{495}$

   (c) $\frac{1}{165}$

   (d) $\frac{2}{165}$

   (e) $\frac{299}{990}$

7. $\sqrt[3]{x^2 \sqrt{x^3}}$ simplifies to

   (a) x

   (b) $x^{7/6}$

   (c) $x^{13/6}$

   (d) $x^{5/6}$

   (e) 1

8. $(1 + i)^{10}$ is equal to

   (a) $32(1 + i)$

   (b) $-32i^2$

   (c) $32i$

   (d) $1 + i^{10}$

   (e) none of the above

9. The solution set for the inequality $|2x + 1| > 0$ is

   (a) the set of all real numbers

   (b) $\{x | x < -\frac{1}{2}\}$

   (c) $\{x | x > -\frac{1}{2}\}$

   (d) $\{x | x \neq -\frac{1}{2}\}$

   (e) none of the above

10. The solution set for $2 \log_5 \sqrt{x} = 3$ is

   (a) $\{0\}$

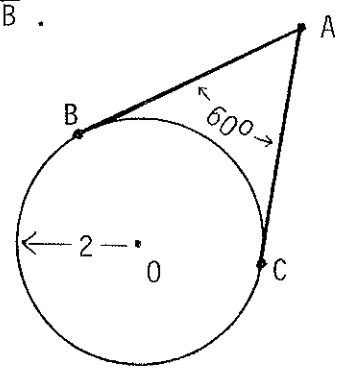
   (b) $\{5\}$

   (c) $\{25\}$

   (d) $\{125\}$

   (e) $\emptyset$

11. In a plane, $\overline{AB}$ and $\overline{AC}$ are tangents to a circle with center O, radius 2. If m $\angle BAC = 60^o$, find the length of $\overline{AB}$.



   (a) $2\sqrt{2}$

   (b) 3

   (c) $2\sqrt{3}$

   (d) 4

   (e) none of the above

12. If the surface area of a cube is 216 m^2 then the volume of a sphere inscribed in the cube would be

   (a) $(360 \cdot \pi)$ kl.

   (b) $(36 \cdot \pi)$ kl.

   (c) (360) kl.

   (d) (3600) kl.

   (e) $(36\pi^3)$ kl.

13. The value printed by the program,

```
10  LET  A = 1
20  LET  B = 2
30  FOR  I = 1  TO  5
40  LET  A = B * A
50  LET  B = A/B
60  NEXT  I
70  PRINT  B
80  END
```

is

(a) 2

(b) 4

(c) 8

(d) 16

(e) 32

14. Let $f(x) = 2x^3 - 3x^2 + 2$ for $0 \leq x \leq 2$. What is the maximum value of $f(x)$?

(a) 2

(b) 1

(c) 3

(d) 7

(e) none of the above

15. The seven scores on a test were 99, 95, 95, 94, 93, 92, and 90. The mode of this set of scores is

(a) 90

(b) 93

(c) 94

(d) 95

(e) 99

16. The graph of the polar equation $r = 2B \sin \theta + 2A \cos \theta$ is

(a) a circle with center (A,B) and radius A + B

(b) an ellipse with x-intercepts $(\pm A, 0)$ and y-intercepts $(0, \pm B)$

(c) the graph of the cartesian equation $y = 2B \sin x + 2A \cos x$

(d) a 4-lobed lemniscate with center (A,B).

(e) a circle with center (A,B) and radius $\sqrt{A^2 + B^2}$

17. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} =$

(a) csc 2x

(b) 2 csc x

(c) $\dfrac{1}{2 \sin x}$

(d) 1

(e) Cannot be simplified

18. The solution set for the inequality $|x^2 - 2| > 1$ is

(a) $\{x | -1 < x < 1\}$

(b) $\{x | -\sqrt{3} < x < \sqrt{3}\}$

(c) $\{x | -1 < x < 1 \text{ or } x < -\sqrt{3} \text{ or } x > \sqrt{3}\}$

(d) $\{x | 1 < x < \sqrt{3}\}$

(e) the set of all real numbers

19. A man can row 1 mile upstream in the same amount of time it takes him to row 2 miles downstream. If the rate of the current is 3 miles per hour, how fast can he row in still water?

(a) 6 miles per hour

(b) 3 miles per hour

(c) 9 miles per hour

(d) 1 mile per hour

(e) 8 miles per hour

20. If the circumference of a circle is 4 cm. then the area of any inscribed square must be

(a) $\frac{8}{\pi^2}$ cm^2

(b) $\frac{4}{\pi}$ cm^2

(c) $\frac{2\sqrt{2}}{\pi}$ cm^2

(d) $\frac{4}{\pi^2}$ cm^2

(e) none of the above

21. Let $f(x) = x^2 + 5x + 6$. Someone tells you that he believes $f(x)$ has a root at $x = 1$. Quickly checking, you find that he is wrong, but using one step of the Newton-Raphson method with his guess as a start, you provide a new estimate of the root as

(a) 0

(b) -2

(c) $\frac{-12}{7}$

(d) $\frac{-5}{7}$

(e) -3

22. What is the equation of the tangent line to the curve $y = x^2 - 3x$ at the point $(1, -2)$?

(a) $y = x - 1$

(b) $y = -x - 1$

(c) $y = x + 1$

(d) $y + 2x = 3$

(e) $y - 2x = 3$

23. Given matrices A, 2 by 3; B, 4 by 2; C, 3 by 2; D, 4 by 4; and E, 2 by 4; the size of the matrix $X = AC + EDB$ is

(a) 4 by 3

(b) 3 by 4

(c) 2 by 4

(d) 4 by 2

(e) none of the above

24. $\lim_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$ is equal to

(a) 0

(b) ∞

(c) $\frac{1}{9}$

(d) $\frac{1}{6}$

(e) $\frac{1}{3}$

25. The angle between the vectors $x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$ is

(a) 90^{o}

(b) 30^{o}

(c) 45^{o}

(d) 0^{o}

(e) none of the above

26. The integers 1, 2, 3, 4, 5 are arranged in a random order. The probability that the digits 1, 2, 3 appear next to each other (not necessarily in order) is

(a) $\frac{1}{120}$

(b) $\frac{1}{40}$

(c) $\frac{1}{10}$

(d) $\frac{3}{14}$

(e) $\frac{3}{10}$

27. A straight line L intersects the parabola $y = x^2$ in two points P and Q. The midpoint of the line segment PQ has x-coordinate 3/2. Find the slope of L.

(a) 1

(b) 2/3

(c) 2

(d) 3

(e) 4/3

28. Find an equation of the circle which is tangent to both the x- and y-axes and has its center in the first quadrant on the parabola $y^2 - 4x + 4 = 0$.

(a) $(x - 2)^2 + (y - 2)^2 = 4$

(b) $(x - 1)^2 + (y - 1)^2 = 1$

(c) $(x - 4)^2 + (y - 4)^2 = 16$

(d) $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

(e) $x^2 + y^2 = 1$

29. Find an equation which is satisfied by all points whose distance from the line $y = 1$ is 3 times the distance from the origin.

(a) $y - 1 = 3\sqrt{x^2 + y^2}$

(b) $9x^2 + 8y^2 + 2y - 1 = 0$

(c) $(1 - y)^2 = 3(x^2 + y^2)$

(d) $9x^2 + 9y^2 + 2y - 1 = 0$

(e) $\sqrt{y^2 - 1} = 3\sqrt{x^2 + y^2}$

30. The solution set for $\arccos (2x^2 - 1) = 2 \arccos \frac{1}{2}$ is

(a) $\{+\frac{1}{2}\}$

(b) $\{-\frac{1}{2}\}$

(c) $\{\pm\frac{1}{2}\}$

(d) $\emptyset$

(e) none of the above

31. $\dfrac{2\cos 190^\circ + 2i \sin 190^\circ}{\cos 70^\circ + i \sin 70^\circ}$ is equal to

(a) 6

(b) $-1 + i\sqrt{3}$

(c) $\sqrt{3} - 1$

(d) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

(e) none of the above

32. The solution set for $\sqrt[3]{x + 1} = \sqrt[6]{3x + 7}$ is

(a) $\{3, -2\}$

(b) $\{3\}$

(c) $\{-2\}$

(d) $\{-3, 2\}$

(e) $\emptyset$

33. In $\triangle ABC$, $AB = 14$, $BC = 15$, $AC = 13$; find the length of the altitude CD.

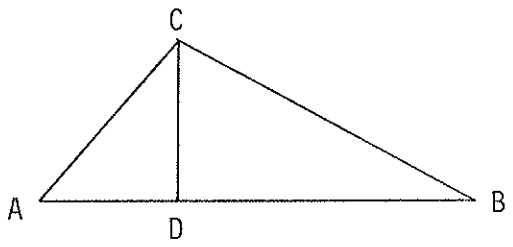
(a) 10

(b) $\dfrac{13.15}{14}$

(c) 11

(d) 12

(e) $\dfrac{14}{13.15}$



34. Which of the following is <u>false</u>? The equation $x^4 - 8x^2 + 4 = 0$ has

(a) two positive roots

(b) two negative roots

(c) no imaginary roots

(d) no rational roots

(e) no irrational roots

35. Which of the following is a sufficient condition for a triangle to be equilateral?

(a) The altitudes intersect at the orthocenter.

(b) The centroid and the circumcenter coincide.

(c) The incenter is equidistant from the three sides.

(d) The nine-point circle passes through the feet of the altitudes.

(e) None of the above.

36. The nonlinear systems of equations $\begin{cases} x^2 - y^2 = c \\ \frac{x^2}{4} + y^2 = 1 \end{cases}$ has

(a) no real solutions if $c < 4$

(b) four real solutions if $c = 4$

(c) four real solutions if $0 \leq c < 4$

(d) two real solutions if $c > 4$

(e) none of the above

37. Find the numbers a and b if the curve with equation $y = ax^2 + bx + 7$ is tangent to the line with equation $y = -5x + 11$ at the point $(1, 6)$.

(a) $a = -2, b = 1$

(b) $a = 1, b = -7$

(c) $a = -4, b = 3$

(d) $a = 2, b = -3$

(e) $a = 0, b = -5$

38. Car A is moving north from point O at 60 miles per hour and Car B is moving east from O at 50 miles per hour. How fast is the distance between them changing when A is 4 miles from O and B is 3 miles from O?

(a) 110 miles per hour
(b) 78 miles per hour
(c) 76 miles per hour
(d) 55 miles per hour
(e) $\sqrt{6100}$ miles per hour

39. The determinant of the matrix

$$\begin{bmatrix} \alpha^2 & (\alpha+1)^2 & (\alpha+2)^2 & (\alpha+3)^2 \\ \beta^2 & (\beta+1)^2 & (\beta+2)^2 & (\beta+3)^2 \\ \gamma^2 & (\gamma+1)^2 & (\gamma+2)^2 & (\gamma+3)^2 \\ \delta^2 & (\delta+1)^2 & (\delta+2)^2 & (\delta+3)^2 \end{bmatrix}$$

is equal to

(a) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(b) 1

(c) 0

(d) $\alpha^2 + (\beta+1)^2 + (\gamma+2)^2 + (\delta+3)^2$

(e) none of the above

40. A right circular cone is cut by a plane parallel to its base and bisecting its altitude. The volume of the smaller cone which is formed is what part of the volume of the original cone?

(a) $\frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{1}{8}$

(d) $\frac{1}{3}$

(e) none of the above