TWENTIETH ANNUAL MATHEMATICS CONTEST
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ADVANCED TOPICS TEST

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Scoring Formula: 4R - W

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This test was prepared from a list of Advanced Topics questions submitted by Carson Newman College.

DIRECTIONS:

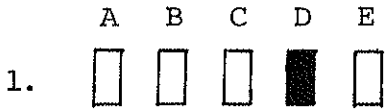
Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers; one and only one is correct. You are to work each problem, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. You must use a pencil with soft lead (No. 2 lead or softer). A sample problem follows:

1. If $2x = 3$, then x equals

(a) 2/3. (b) 3. (c) 6.

(d) 3/2. (e) none of these



The correct answer for the sample problem is 3/2, which is answer (d); so you would answer this problem by making a heavy black mark under space D as indicated above.

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers so it is advisable not to do much wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

The answer sheets will be used for a statistical compilation and will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 1 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. $\log_2 4 =$

   (a) 16

   (b) 2

   (c) 1/2

   (d) 8

   (e) 4

2. If $x < -2$, which of the following is equal to $|2+x|$?

   (a) $-2 + x$

   (b) $2 + x$

   (c) $-2 - x$

   (d) $2 - x$

   (e) none of these

3. If $f(x) = 6x^4 - 2x^3 - x^2 + 9$, then $f'(-2) =$

   (a) 164

   (b) -220

   (c) -212

   (d) -164

   (e) none of these

4. The value of the determinant $\begin{vmatrix} 1 & 0 & -2 \\ 3 & 5 & -1 \\ 4 & 2 & -3 \end{vmatrix}$ is:

   (a) 15

   (b) -69

   (c) 11

   (d) 39

   (e) none of these

5. The domain of the function $y = f(x) = \log_{10} x$ is:

   (a) all real numbers

   (b) all positive real numbers

   (c) all non-negative real numbers

   (d) all real numbers greater than -1

   (e) none of these

6. The repeating decimal $.1\overline{23}$ may be represented as:

   (a) 123/1000

   (b) 12323/100,000

   (c) 245/980

   (d) 61/495

   (e) none of these

7. The expression $1 - \frac{\cos^2\theta}{1 + \sin\theta}$, $0 < \theta < \pi$, is identical with:

   (a) $\cot\theta$

   (b) $\csc\theta$

   (c) $\sin\theta$

   (d) $\cos\theta$

   (e) $\tan\theta$

8. What is the maximum value of the function defined by $f(x) = x^3 + x^2$, $-2 \leq x \leq 1$?

   (a) -2

   (b) 0

   (c) 4/27

   (d) 20/27

   (e) 2

9. In two successive tosses of a coin, the probability of getting one head and one tail is:

(a) 0

(b) 1/4

(c) 1/3

(d) 1/2

(e) 1

10. The price of an article was increased by x%. Later the new price was decreased by x%. If the final price is one dollar, the original price in dollars was:

(a) $\frac{1-x^2}{200}$

(b) $\frac{\sqrt{1-x^2}}{100}$

(c) 1

(d) $1 - \frac{x^2}{10,000-x^2}$

(e) $\frac{10,000}{10,000-x^2}$

11. For the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $A^3 =$

(a) A

(b) A^2

(c) -A

(d) I

(e) none of these

12. If $(0,0)$, (a,b), and (c,d) are collinear, then

(a) $ad - bc = 0$

(b) $ac - bd = 0$

(c) $ad + bc = 0$

(d) $ac + bd = 0$

(e) $a = c = 0$ or $b = d = 0$

13. If $x > 10$ and $x^{3\log_{10} x} = 10^{75}$, then $\log_{10} x =$

(a) 3

(b) 5

(c) $5\sqrt{3}$

(d) 25

(e) 25/3

14. Which of the following is NOT an equivalent expression for $\cos 2x$:

(a) $\cos^2 x - \sin^2 x$

(b) $\cos^4 x - \sin^4 x$

(c) $1 - 2\cos^2 x$

(d) $\dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

(e) $1 - 2\sin^2 x$

15. If θ is the acute angle for which $\sin \theta = 4/5$, then $\sin 2\theta =$

(a) 8/5

(b) 9/25

(c) 16/25

(d) 24/25

(e) 9/5

16. If the intersection of the graphs $y = 2x^2 - 4x + 5$ and $y = c$ contains one and only one point, then $c =$

(a) -11

(b) 0

(c) 5

(d) 3

(e) 11

17. $\text{Log}_{10}p + \text{Log}_{10}q = \text{Log}_{10}(p + q)$ only if:

(a) $p = q = 0$

(b) $p = \frac{q^2}{1 - q}$

(c) $p = q = 1$

(d) $p = \frac{q}{q - 1}$

(e) $p = \frac{q}{q + 1}$

18. The graphs of $y = \frac{x^2 - 9}{x - 3}$ and $y = 2x$ intersect in:

(a) one point whose abscissa is 3

(b) one point whose abscissa is 0

(c) no points

(d) two distinct points

(e) none of these

19. A pair of dice is thrown. The probability that the sum of the two numbers appearing on the dice is a prime number is:

(a) 5/12

(b) 11/36

(c) 7/18

(d) 5/18

(e) none of these

20. Which of the following is a term in the expansion of $(a + b)^{10}$:

(a) $210a^{10}$

(b) $210a^5b^2$

(c) $210\ ab^{10}$

(d) $210a^6b^4$

(e) $210a^5b^5$

21. In triangle ABC, AB + BC = 14, BC + AC = 16, AB + AC = 18. What kind of triangle is ABC?

(a) isosceles

(b) equilateral

(c) right

(d) right and isosceles

(e) none of these

22. Let A, B, and C be the angles of a triangle and let a,b,c be the respective opposite sides; if a = 4, b = 5, and c = 2, then cos B is

(a) 5/16

(b) -5/16

(c) 4/5

(d) 5/4

(e) none of these

23. If $\log_{10}(x + 4) + \log_{10}(x + 1) = 1$, then x =

(a) -2

(b) -6

(c) 5

(d) 1

(e) 3

24. The multiplicative inverse, A^{-1}, of the matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$ is:

(a) $\begin{pmatrix} 0 & -1/3 \\ 1 & 2/3 \end{pmatrix}$

(b) $\begin{pmatrix} 2/3 & -1/3 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} -2/3 & 1/3 \\ 3 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1/3 \\ -1 & 2/3 \end{pmatrix}$

(e) none of these

25. If a sample space consists of exactly two points with probabilities x and x^2, then x is:

(a) $1/4$

(b) $\sqrt{3}/4$

(c) $1/2$

(d) $\dfrac{\sqrt{5} - 1}{2}$

(e) $\sqrt{2}/2$

26. If a and b are positive numbers, which of the following are necessarily true?

(I) $\sqrt{ab} \leq \dfrac{a + b}{2}$ (II) $\dfrac{1}{a} + \dfrac{1}{b} \leq \dfrac{a + b}{2}$ (III) $\dfrac{2}{\frac{1}{a} + \frac{1}{b}} \leq \dfrac{a + b}{2}$

(a) I only

(b) I and II only

(c) I and III only

(d) II and III only

(e) I, II, and III

27. A "mini" deck of cards consists of only three cards. The cards are colored in this manner: card number one has 1 blue and 1 white face, card number two has 2 blue faces and card number 3 has 2 white faces. A card is drawn at random and placed flat on the table. The face showing is white. The probability that the other side is white is

(a) 1/4

(b) 1/3

(c) 1/2

(d) 2/3

(e) 3/4

28. The quotient resulting when 6 is divided by $3(\cos 120^\circ + i \sin 120^\circ)$ is

(a) $2(\cos 120^\circ + i\sin 120^\circ)$

(b) $3(\cos 20^\circ + i\sin 20^\circ)$

(c) $\cos 60^\circ - i\sin 60^\circ$

(d) $2(\cos 120^\circ - i\sin 120^\circ)$

(e) $2(\cos 60^\circ + i\sin 60^\circ)$

29. The sum of the roots of $x^2 + bx + c = 0$ is 5 and their product is 6. What is b?

(a) 3

(b) 5

(c) 6

(d) -5

(e) -6

30. If $f(x) = \dfrac{x^2 + x}{x + 1}$, $x \neq -1$, how must $f(-1)$ be defined so that f is continuous at $x = -1$?

(a) 0

(b) -1

(c) 2

(d) 1

(e) none of these

31. The diameter of a circle is divided into K line segments which are equal in length. On each line segment, a semicircle is constructed. The limit as K goes to infinity of the sum of the lengths of the semicircles is:

(a) equal to one-half the circumference of the original circle.

(b) equal to the diameter of the original circle.

(c) greater than the diameter but less than one-half the circumference of the original circle.

(d) infinite

(e) greater than one-half the circumference, but finite.

32. The equation of the circle with center at (5,3) and tangent to the line $y = x$ is:

(a) $(x - 5)^2 + (y - 3)^2 = \sqrt{8}$

(b) $(x - 3)^2 + (y - 5)^2 = \sqrt{3}$

(c) $(x - 3)^2 + (y - 5)^2 = 5$

(d) $(x - 5)^2 + (y - 3)^2 = 2$

(e) none of these

33. What is the probability of drawing a flush (5 cards of the same suit) from a regular deck of playing cards (without placement)?

(a) 5/13

(b) 5!/52!

(c) 120/2563

(d) 33/16,660

(e) none of these

34. $(1 + i\sqrt{3})^6 =$

(a) 1

(b) 1 + 27i

(c) $1 + i\sqrt{3} + 3i^2$

(d) i^6

(e) none of these

35. If $m > 0$ and the area of the region enclosed by the graphs of $y = mx$ and $y = x^2$ is 36, then $m =$

(a) 216

(b) $36 \cdot 6^{1/3}$

(c) 6

(d) $\frac{36}{6^{1/3}}$

(e) none of these

36. If for all real x, $f(x) = x^2$, $h(x) = f(1 + g(x))$ $\begin{matrix} g'(1) = 1 \\ h'(1) = 1 \end{matrix}$ then $g(1) =$

(a) -1

(b) -1/2

(c) 0

(d) 1/2

(e) 1

37. The value of $\int_{-\pi}^{\pi} |\cos x| dx$ is

(a) 0

(b) $\sin \pi$

(c) $\cos \pi$

(d) 2

(e) 4

38. Two coins of equal denomination are placed on a table so that they are tangent. Two rays from the center of one coin intersect the other coin so that they are tangent to it. Find the area of the region between the coins and the rays in terms of the radius r of each coin.

(a) $r^2(\sqrt{3} - \pi/2)$

(b) $r(\sqrt{3} - r\pi/2)$

(c) $r^2(\sqrt{5} - \pi/2)$

(d) $r^2(\sqrt{3} - \pi)$

(e) none of these

39. The lengths of the medians which are drawn from the vertices of the acute angles of right triangle XYZ are 5m and $\sqrt{40}$ m. The length of the hypotenuse of triangle XYZ is:

(a) $2\sqrt{40}$ m

(b) $2\sqrt{13}$ m

(c) 10 m

(d) $\sqrt{13}$ m

(e) impossible to determine from the given information.

40. Assuming that the following Y_k values belong to a polynomial of degree 4, compute the next 3 values.

k	0	1	2	3	4	5	6	7
Y_k	1	0	2	1	0	-	-	-

(a) $Y_5 = 2$, $Y_6 = 1$, $Y_7 = 0$

(b) $Y_5 = 11$, $Y_6 = 44$, $Y_7 = 107$

(c) $Y_5 = 9$, $Y_6 = 12$, $Y_7 = 11$

(d) $Y_5 = 11$, $Y_6 = 55$, $Y_7 = 162$

(e) none of these