NINETEENTH ANNUAL MATHEMATICS CONTEST
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ADVANCED TOPICS TEST

1975

Scoring Formula: 4R - W

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This test was prepared from a list of Advanced Topics questions submitted by Tennessee Technological University.

DIRECTIONS:

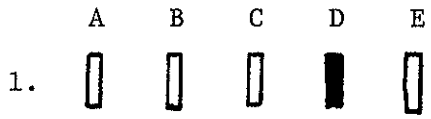
Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers; one and only one is correct. You are to work each problem, determine the correct answer, and indicate your choice by making a heavy black mark in the correct place on the separate answer sheet provided. You must use a pencil with soft lead (No. 2 lead or softer). A sample problem follows:

1. If $2x = 3$, then x equals

   (a). 2/3. (b). 3. (c). 6.

   (d). 3/2. (e). none of these



The correct answer for the sample problem is 3/2, which is answer (d); so you would answer this problem by making a heavy black mark under space D as indicated above.

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers so it is advisable not to do much wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet.

The answer sheets will be used for a statewide statistical compilation and will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 1 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

1. The repeated decimal $.8\overline{27}$ may be represented as

   (a) $\frac{82727}{100000}$

   (b) $\frac{9}{11}$

   (c) $\frac{91}{110}$

   (d) .82727

   (e) none of these

2. The determinant $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ is equal to

   (a) $(y - x)(z - x)(z - y)$

   (b) $(x - y)(z - x)(z - y)$

   (c) $(y - x)(x - z)(z - y)$

   (d) $(y - x)(z - x)(y - z)$

   (e) none of these

3. There are two daily newspapers in a certain town, paper A and paper B. A survey shows that 900 people read either paper A or paper B or both. If 200 people read both A and B, and 400 people read paper B, how many people read paper A?

   (a) 500

   (b) 700

   (c) 300

   (d) 400

   (e) none of these

4. Let α be an angle whose radian measure is $-\frac{5\pi}{4}$. Then the degree measure of α is

   (a) 135°

   (b) -392°

   (c) -225°

   (d) 236°

   (e) none of these

5. Which of the following numbers are in the range of the function defined by $f(x) = \frac{2}{3} \sec x$?

(a) $-\frac{3}{4}$

(b) 0

(c) $-\frac{1}{4}$

(d) $\frac{1}{2}$

(e) none of these

6. The value of $\lim_{t \to 1} \frac{t - 1}{|1 - t|}$ is

(a) 1

(b) -1

(c) 0

(d) does not exist

(e) none of these

7. The inverse A^{-1} of the matrix $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$ is

(a) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & -\frac{1}{2} \end{pmatrix}$

(b) does not exist

(c) $\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$

(e) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$

8. The coordinates of the midpoint of the line segment joining the two points $(-1, 5)$ and $(3, -1)$ are

   (a) $(1, 2)$

   (b) $(2, -3)$

   (c) $(-2, 3)$

   (d) $(2, 1)$

   (e) none of these

9. The algorithm for approximating $\sqrt{2}$ by using Newton's method of finding roots of $f(x) = 0$ is

   (a) $x_{i+1} = x_i - \frac{1}{x_i} \qquad i = 0,1,2,\ldots$

   (b) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{2}{x_i}\right) \quad i = 0,1,2,\ldots$

   (c) $x_{i+1} = \frac{1}{2}\sqrt{x_i} \qquad i = 0,1,2,\ldots$

   (d) $x_{i+1} = \frac{1}{2}\left(x_i - \frac{2}{x_i}\right) \quad i = 0,1,2,\ldots$

   (e) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{1}{x_i}\right) \quad i = 0,1,2,\ldots$

10. A student is to answer 8 out of 10 questions on an examination. The number of different ways 8 questions out of 10 can be answered is

    (a) 90

    (b) $\frac{10!}{2!}$

    (c) 45

    (d) $\frac{10!}{8!}$

    (e) $(10)^8$

11. If one zero of the polynomial $2x^3 - kx^2 - 5x + 2$ is the reciprocal of another then k equals

(a) 2

(b) 5

(c) -2

(d) -5

(e) none of these

12. The number of diagonals of a convex polygon of 14 sides is

(a) 42

(b) 154

(c) 11

(d) 77

(e) 28

13. Which of the following is an identity?

(a) $\tan \frac{x}{2} = \frac{1 - 2 \cos x}{\sin x}$

(b) $\cos^4 x - \sin^4 x = \cos 2x$

(c) $\frac{1 + \sec x}{\sec x} = \frac{\cos x}{1 + \cos x}$

(d) $\cos 4x = \cos^4 x - \sin^4 x - 2 \sin^2 x \cos^2 x$

(e) none of these

14. Three points A, B, and C form a triangle with angles α, β, and γ respectively. If a, b, and c are the lengths of the sides opposite to α, β, and γ respectively and if the measure of $\alpha = 45°$, the measure of $\beta = 30°$ and $a = 10$ in miles then b, in miles, is

(a) $\frac{10}{\sqrt{3}}$

(b) 10

(c) 33

(d) $5\sqrt{2}$

(e) none of these

15. For the diagonal matrix $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ to possess an inverse it is necessary and sufficient that

(a) $a_1 a_2 a_3 \neq 0$

(b) $a_1^2 + a_2^2 + a_3^2 > 0$

(c) the diagonal elements are all positive

(d) the diagonal elements are all negative

(e) any of these

16. An equation of the parabola with vertex at (2, 4) and focus at (-3, 4) is

(a) $y^2 - 20x - 8y + 56 = 0$

(b) $y^2 + 20x - 8y - 24 = 0$

(c) $x^2 - 4x - 20y + 84 = 0$

(d) $y^2 + 20x - 8y + 76 = 0$

(e) none of these

17. The vertex of the parabola $x^2 + 4x + y + 3 = 0$ is

(a) $(2, -1)$

(b) $(-2, -1)$

(c) $(2, 1)$

(d) $(-2, 1)$

(e) none of these

18. The interpolating polynomial of degree two for the data:

x_i	0	1	2
$f(x_i)$	1	3	7

is

(a) $2x^2 + 1$

(b) $x^2 + x - 1$

(c) $x^2 - x + 1$

(d) $x^2 + 2x$

(e) $x^2 + x + 1$

19. Students in a class are selected at random, one after the other, from a class consisting of 4 boys and 3 girls. The probability that boys and girls in the class alternate starting a boy first is

(a) $\frac{1}{42}$

(b) $\frac{1}{35}$

(c) $\frac{2}{\binom{7}{3}}$

(d) $\frac{4}{7}$

(e) $\frac{3}{7}$

20. If x and y are both unequal positive numbers, the value of

$$\left| \frac{|x - y| - |x + y|}{2} \right|$$ is

(a) x

(b) y

(c) $x + y$

(d) the smaller of x and y

(e) none of these

21. A polynomial with integral coefficients $P(x)$ is divided by $x - r$ until a remainder R_1 free of x is obtained; then the remainder theorem states that $R = P(r)$; ie., $P(x) = (x - r) \cdot Q(x) + P(r)$. Suppose that $P(1) = 6$. Which of the numbers $\{1,2,3,6\}$ cannot be solutions of the equation $P(x) = 0$?

(a) 1 and 2

(b) 1 and 3

(c) 2 and 3

(d) 1 and 6

(e) 2 and 6

22. The solution set, where $0 \leq x < 2\pi$, for $2 \sin^2 (2x) - 1 = 0$, is

(a) $\{\frac{\pi}{4}, \frac{5\pi}{4}\}$

(b) $\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$

(c) $\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}\}$

(d) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

(e) none of these

23. Let A, B, C be square matrices of order n and let z be the zero square matrix of order n. If AB = AC then

(a) B = C

(b) A = Z

(c) B = C = Z

(d) A is non-singular

(e) none of these

24. An equation of the straight line through the point (2, 1) and perpendicular to the line 2y + x = 1 is

(a) x + 2y = 1

(b) y = 2x

(c) y = 2x - 3

(d) x + 26 = y

(e) none of these

25. An equation of the circle having intercepts of 3 and 5 on the x - and y - axes respectively and passing through the origin is

(a) $x^2 + y^2 + 3x + 5y - 18 = 0$

(b) $x^2 + y^2 - 5x + 3y - 18 = 0$

(c) $x^2 + y^2 - 5x - 3y = 0$

(d) $x^2 + y^2 - 3x - 5y = 0$

(e) none of these

26. An approximation for $\int_1^3 f(x)\,dx$ using Simpson's rule for the data:

x_i	1	2	3
$f(x_i)$	1	$\frac{1}{2}$	$\frac{1}{3}$

is

(a) $\frac{10}{9}$

(b) $\frac{3}{2}$

(c) 1

(d) ln 3

(e) 10

27. A pair of dice is thrown. Then the probability of the sum of two numbers appearing on the dice being 4 or less is

(a) $\frac{6}{36}$

(b) $\frac{2}{15}$

(c) $\frac{6}{30}$

(d) $\frac{1}{12}$

(e) none of these

28. The value of $\lim_{x \to 1} \frac{3x^4 - 4x^3 + 3x - 5 + 3}{x - 1}$ is

(a) 0

(b) 1

(c) does not exist

(d) 3

(e) none of these

29. Which of the following sets of grades has the same mean and median?

(a) 90, 60, 75, 70, 80

(b) 80, 65, 75, 75, 70

(c) 80, 60, 75, 75, 70

(d) 70, 75, 75, 80, 100

(e) none of these

30. The congruence $9x + 12 \equiv 6 \pmod{15}$ has how many integral solutions ?

(a) no integral solutions

(b) one solution

(c) three solutions

(d) infinitely many solutions

(e) none of the above

31. In how many ways may eight people be seated around a circular table if three of these people insist upon sitting in three adjacent chairs ?

(a) 8!

(b) 7!

(c) 6!

(d) 5!

(e) none of these

32. A sphere 12 inches in diameter made of materials weighing 20 lbs. per cubic foot is floating in a liquid whose density is 128 lbs. per cubic foot. The depth to which the sphere sinks is

(a) 3.4"

(b) 3"

(c) 4.2"

(d) 4"

(e) 3.8"

33. If $\tan x = \frac{1}{2}$ and $5\pi < x < 6\pi$, then the value of $\cos\left(\frac{x}{2}\right)$ is

(a) $-\sqrt{\frac{5 - 2\sqrt{5}}{10}}$

(b) $\sqrt{\frac{5 - 2\sqrt{5}}{10}}$

(c) $-\sqrt{\frac{5 + 2\sqrt{5}}{10}}$

(d) $\sqrt{\frac{5 + 2\sqrt{5}}{10}}$

(e) none of these

34. The number of roots, which are not real, of the equation

$3x^4 + 4x^3 + 2x - 1 = 0$ is

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

35. Let A be an $n \times n$ real matrix whose determinant is denoted by det A. Let A^T denote the transposed matrix of A. If $A = -A^T$ then

(a) det A^T = - det A for all n

(b) det $A = 0$ for all n

(c) det $A = 0$ for odd n

(d) det $A \neq 0$ for all n

(e) none of these

36. An equation of the circle with center at (6, 2) and tangent to the line $y = x$ is

(a) $x^2 + y^2 - 4x - 12y + 32 = 0$

(b) $x^2 + y^2 - 12x - 4y - 32 = 0$

(c) $x^2 + y^2 - 12x - 4y + 36 = 0$

(d) $x^2 + y^2 + 12x + 4y + 32 = 0$

(e) $x^2 + y^2 - 12x - 4y + 32 = 0$

37. All solutions to the system $\begin{cases} 2x + 3y - 2z = 5 \\ x + y - z = 2 \end{cases}$ using Gauss-Jordan reduction method, are of the form

(a) $\bar{x} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$

(b) $\bar{x} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$

(c) $\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(d) $\bar{x} = \begin{pmatrix} c \\ 1 \\ c-1 \end{pmatrix}$, c a real number

(e) none of these

38. Two cards are drawn from an ordinary deck of 52 cards. The probability of drawing two spades is

(a) $\frac{39 \cdot 38}{\binom{52}{2}}$

(b) $\frac{1}{26}$

(c) $\frac{\binom{13}{2}}{52 \cdot 51}$

(d) $\frac{1}{17}$

(e) $\frac{(13)^2}{(52)^2}$

39. How many integral solutions does the equation $x^2 + y^2 = 4z + 3$ have ?

(a) no integral solutions

(b) one solution

(c) 3 solutions

(d) infinitely many solutions

(e) none of the above

40. Hyp I: If I owned all the gold in Fort Knox, then I would be wealthy.

Hyp II: I do not own all the gold in Fort Knox.

If we assume the above hypotheses to be true, which of the following statements can we logically deduce from them ?

(a) I am not wealthy.

(b) I am wealthy.

(c) I own all of the gold in Fort Knox.

(d) I own some of the gold in Fort Knox.

(e) none of the above