FORTY-FIFTH ANNUAL MATHEMATICS CONTEST
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THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II 2001

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Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

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ADVANCED TOPICS II 2001

1. A function $f(x)$ has derivative $f'(x) = x^2 + 4x$ and value $f(1) = 7$. What is the formula for $f(x)$?

   a. $\frac{x^3}{3} + 2x^2$

   b. $\frac{x^3}{3} + 2x^2 + 7$

   c. $x^2 + 4x + 2$

   d. $\frac{x^3}{3} + 2x^2 + \frac{14}{3}$

   e. $x^3 + 2x^2 + 14$

2. At what point on the graph of $f(x) = x^3 - 6x + 4$ is the slope of the tangent line the smallest possible?

   a. $(-6, 0)$ b. $(0, 4)$ c. $(-6, -176)$ d. $(6, 184)$ e. $(4, 0)$

3. Consider the function $f(x) = x^2 - 4x$ defined on the closed interval $[-1, 6]$. The largest and smallest values of $f(x)$ on the given domain are, respectively,

   a. 0 and -4 b. 12 and 5 c. 18 and -10 d. 12 and -4 e. 0 and 4

4. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$ is

   a. 0 b. $\frac{\pi}{4}$ c. $\frac{3\pi}{4}$ d. 1 e. undefined

5. Evaluate: $\int \sec^5 x \tan^3 x \, dx =$

   a. $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

   b. $\frac{1}{24}\sec^6 x \tan^4 x + C$

   c. $\frac{1}{4}\sec^4 x \tan x + C$

   d. $\frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 x + C$

   e. none of these.

6. Which integral represents the area of the region bounded by the graphs of $y = x$ and $y = 5x - x^3$?

   a. $\int_{-2}^{2} (x^3 - 4x)\, dx$

   b. $\int_{-2}^{2} (4x - x^3)\, dx$

   c. $2\int_{0}^{2} (4x - x^3)\, dx$

   d. $\int_{-2}^{0} (4x - x^3)\, dx + \int_{0}^{2} (x^3 - 4x)\, dx$

   e. none of these.

7. Which integral represents the volume of the solid formed by revolving about the line $x = 3$ the region bounded by the graphs of $y = \ln x$, $y = 0$ and $x = 3$?

   a. $\pi \int_1^3 [9 - (\ln x)^2]\, dx$

   b. $\pi \int_0^{\ln 3} [9 - e^{2y}]\, dy$

   c. $\pi \int_0^{\ln 3} (3 - e^y)^2\, dy$

   d. $\pi \int_1^3 (\ln x)^2\, dx$

   e. none of these.

8. The derivative of $3^x x^3$ is

   a. $2x^2 3^{x-1}$

   b. $9x^2$

   c. $3^{x-1}x^2[9 + x^2]$

   d. $3^x x^2[3 + (\ln 3)x]$

   e. none of these.

9. Evaluate: $\int \frac{e^{-5/x}}{x^2}\, dx =$

   a. $\frac{1}{5}xe^{-5/x} + C$

   b. $\frac{1}{5}e^{5/x} + C$

   c. $\frac{1}{5}xe^{5/x} + C$

   d. $\frac{1}{5}e^{-5/x} + C$

   e. none of these.

10. Evaluate: $\int \frac{3x + 4}{(x^2 + 4)(3 - x)}\, dx =$

    a. $\frac{1}{2}\ln(x^2 + 4) + \ln|3 - x| + C$

    b. $\frac{1}{2}\arctan\left(\frac{x}{2}\right) + \ln|3 - x| + C$

    c. $\frac{1}{2}\arctan\left(\frac{x}{2}\right) - \ln|3 - x| + C$

    d. $\ln\left|\frac{\sqrt{x^2 + 4}}{3 - x}\right| + C$

    e. none of these.

11. Evaluate the limit using L'Hôpital's Rule, if applicable: $\lim_{x\to 0} \dfrac{\sin x - x}{x^3} =$

a. 1/6
b. 0
c. $-1/6$
d. The limit doesn't exist.
e. none of these.

12. The infinite series $\sum_{n=3}^{\infty} \dfrac{2}{3^n}$ has as its sum

a. 3 b. 1/3 c. 1/9 d. 4/3 e. none of these

13. The series $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^n}{n!}$

a. diverges.
b. converges conditionally.
c. converges to e^x.
d. converges to $-e^{-x}$.
e. converges to 0.

14. The tangent line to the curve defined by $-48 + 16x + 24y - 8xy - 3y^2 + xy^2 = 0$ at the point $(3, 5)$

a. is horizontal.
b. has slope 1.
c. has slope -1.
d. is vertical.
e. does not exist.

15. Given the region defined by $0 \le y \le -4 + 5x - x^2$, what is the volume of the solid produced when the region is revolved about the y-axis?

a. $\dfrac{555\pi}{2}$ b. $\dfrac{285\pi}{2}$ c. $\dfrac{45\pi}{2}$ d. 9π e. $\dfrac{45\pi}{4}$

16. If $f(x) = \tan x$, what is $f^{(3)}(0)$? That is, what is $\left.\dfrac{d^3 \tan x}{dx^3}\right|_{x=0}$?

a. -2 b. -1 c. 0 d. 1 e. 2

17. Newton's method for finding a root of the equation $f(x) = 0$ says that, given x_0 near a root, the values of the sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ will approach the root. The function $f(x) = x^3 + x - 3$ has a root between $x = 1$ and $x = 3$. Starting with $x_0 = 3$, what is x_4?

a. 1.21341 b. 1.2144 c. 1.25244 d. 1.47945 e. 2.03571

18. A population is growing so that the rate of growth is given by the equation

$$\frac{dy}{dt} = 0.01\, y(t)\left(1 - \frac{y(t)}{1000000}\right).$$

For what value of $y(t)$ is the rate of growth of $\dfrac{dy}{dt}$ largest?

a. 500000 b. 400000 c. 0 d. 600000 e. 1000000

19. Determine the interval $[a, b]$ over which the definite integral $\int_a^b (5x - 4 - x^2)\,dx$ has its maximum value.

a. $[-\infty, \infty]$ b. $[-\infty, 1]$ c. $[1, \infty]$ d. $[1, 4]$ e. $[0, \frac{5}{2}]$

20. The tangent and normal lines to the function $f(x) = e^x$ at $x = 0$ intersect the x-axis to form a right triangle in the first and second quadrants. What is the area of this triangle?

a. 1 b. e c. 2 d. $\sqrt{2}$ e. 1/2

21. The number of non-empty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which contain only even numbers is

a. $9! - 1$
b. $2^9 - 1$
c. $4! - 1$
d. $\binom{9}{4}$
e. $2^4 - 1$

22. For $n = 1, 2, 3 \ldots$ the determinant of the matrix $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}^n$ is

a. $\dfrac{(-1)^n}{2^{2n}}$
b. $\left(\dfrac{1}{4}\right)^n$
c. always negative
d. always positive
e. $\dfrac{1}{2} - 1$

23. Because of poor financial results at his company, George received a 25% pay cut. By what percentage must his new pay rate be raised to bring it back to the original level?

a. 25% b. 50% c. 100% d. $33\frac{1}{3}\%$ e. 40%

24. Cube A has volume V. What is the volume of cube B, whose surface area is twice the surface area of cube A?

a. V^2 b. $\sqrt{2}V$ c. $2^{3/2}V$ d. $2V$ e. $2^{2/3}V$

25. For $a > 1$ and $b > 0$, $\ln a^2 = \log_{10} b^2$ will be true if

a. $a = b$
b. $\ln a \log_{10} e = \log_{10} b$
c. $\ln a \log_{10} b = \ln b$
d. $a = e^{\log_{10} b}$
e. $\ln\left(\frac{a}{b}\right) = 1$

26. An urn contains 6 red balls and 4 black balls. Two balls are drawn at random from the urn without replacement. Find the probability that one ball of each color is drawn.

a. $\dfrac{2}{3}$ b. $\dfrac{21}{45}$ c. $\dfrac{1}{2}$ d. $\dfrac{24}{45}$ e. $\dfrac{1}{5}$

27. A multiple choice test has 4 questions and 4 possible answers for each question. What is the probability that a student who knows nothing will get a score of 75% or more by guessing?

a. 0 b. $\frac{1}{4}$ c. $\frac{13}{256}$ d. $\frac{1}{256}$ e. $\frac{1}{16}$

28. A multiple choice test has 10 questions and 4 possible answers for each question. George knows the answers to 7 of the questions but must guess at the others. What is the probability that he will make a grade of 90% or better?

a. $\frac{5}{32}$ b. $\frac{1}{3}$ c. $\frac{5}{64}$ d. $\frac{3}{32}$ e. $\frac{5}{16}$

29. Events A and B are independent with $P(A) = .25$ and $P(B) = .5$. What is the probability that either A or B will occur?

a. .625 b. .75 c. .125 d. .5 e. .375

30. The letters of the word Tennessee are on blocks. How many distinguishable sequences of the blocks can be made where each block is used and placed with its letter in its readable position?

a. 3780 b. 362880 c. 15120 d. 126 e. 1

31. An experiment consists of choosing with replacement an integer at random from the numbers 1 to 9 inclusive. If we let M denote that the number is an integral multiple of 3 and N denote that it is not, which of the following sequences of results is least likely?

a. $MNNMN$ b. $NMMN$ c. $NMMNM$ d. $NNMN$ e. $MNMM$

32. The Klingons plan to attack a Cardassian space station with a succession of war birds until it is destroyed. If the Klingons estimate that the probability is .25 that each of their war birds will destroy the space station with an attack, the probability that the fourth war bird sent will be the one to destroy it is closest to? You may assume that each attack is independent of each other attack.

a. 1 b. .3164 c. .2275 d. .22753 e. .1055

33. $\lim_{x \to 1^+} x^{\frac{1}{1-x}}$ is

a. 0 b. .36788 c. 1 d. e^{-1} e. .368

34. Let $y = f(x)$ be a function with an inverse which has properties satisfying the table at left. Complete the table for the inverse function at right.

x	y	$y'(x)$	$y''(x)$
3	9	6	2

y	x	$x'(y)$	$x''(y)$
9	3	1/6	

a. $\frac{1}{36}$ b. $-\frac{1}{36}$ c. $\frac{1}{72}$ d. $\frac{1}{108}$ e. $-\frac{1}{108}$

35. The following computation has steps numbered **1**, **2**, **3** and **4** with the numbers over the equal sign of the step.

$$\int_{-1}^{1} x^{-2}\,dx \overset{\mathbf{1}}{=} \frac{x^{-1}}{-1}\Big|_{-1}^{1} \overset{\mathbf{2}}{=} -\frac{1}{1} - \left(-\frac{1}{-1}\right) \overset{\mathbf{3}}{=} -1 - 1 \overset{\mathbf{4}}{=} -2$$

a. Step **1** is incorrect.
b. Step **2** is incorrect.
c. Step **3** is incorrect.
d. Step **4** is incorrect.
e. All steps are correct.

36. Of all parabolas which pass through $(0,0)$ and $(1,p)$, $p>1$ and which open downward, the one which bounds the smallest area above the horizontal axis is the one which passes through

a. $(2,0)$ b. $\left(\frac{3}{2},0\right)$ c. $(p,0)$ d. $\left(1+\frac{p}{2},0\right)$ e. $(1+p,0)$

37. Which of the functions $f(x)$ below satisfy $\int_0^x f(t)\,dt = \Big(f(x)\Big)^2$?

a. 1 b. e^x c. $\ln x$ d. x^2 e. $x/2$

38. Let $[\![x]\!]$ denote the greatest integer (step) function and let n be a natural number. Then $\int_0^n [\![x]\!]\,dx =$

a. $\frac{(n-1)n}{2}$ b. undefined c. $n!$ d. $(n-1)!$ e. $\left[\!\left[\frac{n^2}{2}\right]\!\right]$

39. Consider a point $P=(a,b)$ in the first quadrant on the graph of the function $f(x)=\frac{1}{x}$. The tangent and normal lines to f at P form a right triangle with hypotenuse along the x-axis. If we let A denote the area of this triangle, what is $\lim_{a\to\infty} A$?

a. 0 b. $\frac{1}{2}$ c. ∞ d. 1 e. e

40. An n foot by n foot area is to be tiled by placing one foot square tiles in place at random. Two of the tiles are red and the rest are green. What is the probability that the two red tiles will share an edge?

a. $\frac{1}{n!}$
b. $\frac{2}{n(n+1)}$
c. $\frac{4}{n(n+1)}$
d. $\frac{6}{n(n+1)}$
e. $\frac{1}{2^n}\binom{n}{2}$