FORTY-THIRD ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II 1999

Prepared by:

Dept. of Mathematics
University of Tennessee-Knoxville

Coordinated by: Ed Clark

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, TN 37044

Scoring formula: 4R – W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State Community College, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
TRW Commercial Steering Division, Lebanon, Tennessee
Wright Industries, Inc., Nashville, Tennessee

ADVANCED TOPICS II 1999

1. Let i be the usual complex number. The exact value of $(1+i)^{1000000}$ is

a) -1000000 b) 2^{1000} c) 2^{500000} d) 2 e) 0

2. Let X be the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then X^{9999} is equal to

a) $\begin{bmatrix} 0 & -9999 \\ 9999 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 9999 \\ -9999 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 2^{9999} & 0 \\ 0 & 2^{-9999} \end{bmatrix}$ e) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

3. $\sqrt{2}+\sqrt{3}$ is a root of which polynomial?

a) $(x^2-2)+(x^2-3)$ b) $(x^2-2)(x^2-3)$ c) x^4-6x^2+5 d) x^4-4x^2+9 e) x^4-10x^2+1

4. A fair coin is tossed four times. What is the probability of obtaining exactly two heads?

a) $\frac{1}{2}$ b) 1 c) $\frac{3}{8}$ d) $\frac{3}{4}$ e) $\frac{9}{16}$

5. $\lim_{x\to 0} \frac{\sin|x|}{|x|} =$

a) -1 b) 1 c) ∞ d) 0 e) does not exist

6. $\lim_{n\to\infty} \frac{e^{\sqrt{n+1}}}{e^{\sqrt{n}}} =$

a) ∞ b) 1 c) e d) $\sqrt{e}$ e) does not exist

7. $\int_0^1 \arctan x\,dx =$

a) tan (1) b) $\frac{\pi}{2}-\frac{1}{3}\ln(2)$ c) $\frac{\pi}{4}+\frac{1}{2}\ln(2)$ d) $\frac{\pi}{4}-\frac{1}{2}\ln(2)$ e) $\frac{\pi}{4}$

8. $\frac{d}{dx}(x^x)=$

a) $(\ln x)^x$ b) $(\ln x+1)x^x$ c) xx^{x-1} d) $(\ln x)^x+1$ e) $(\ln x+1)x^{x-1}$

9. The function $f(x)=x^3-6x^2+kx-8$ has a stationary value (or critical point), but has no local maximum or minimum. The value of k is

a) -12 b) -4 c) 4 d) 12 e) 24

10. The Earth has a diameter of 8000 miles. At a height of 100 miles above the surface of the Earth, the distance to the horizon is

a) 90 miles b) 3900 miles c) 900 miles d) $\sqrt{800000}$ miles e) 400 miles

11. The area enclosed between the curves $y = x^2$ and $x = y^2$ is

a) $\frac{1}{3}$ b) $\frac{1}{12}$ c) $\frac{1}{2}$ d) 1 e) $\frac{5}{12}$

12. $\int_1^2 \ln(x)\,dx =$

a) $-\frac{1}{2}$ b) ln(2) c) e^2-e d) ln(4) - 1 e) ln(4)

13. The area enclosed by the curve, defined in polar coordinates by $r = 1 + \cos(\ominus)$ $(0 \le \ominus < 2\pi)$, is

a) $\frac{3\pi}{2}$ b) 2 c) 2π d) $\pi + 1$ e) $2(\pi - 1)$

14. The normal to the curve $y = \sin x$ at the point $\left(\frac{\pi}{4}, \sin\left(\frac{\pi}{4}\right)\right)$ meets the x-axis at $x =$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}-\frac{1}{\sqrt{2}}$ c) $\frac{\pi}{4}+\frac{1}{2}$ d) 1 e) 0

15. Let $A = (1, 1)$, $B = (2, 2)$. The set of all points P in the Cartesian plane such that $|PA| + |PB| = 2$ is

a) a single point b) a hyperbola c) two points d) an ellipse e) a circle

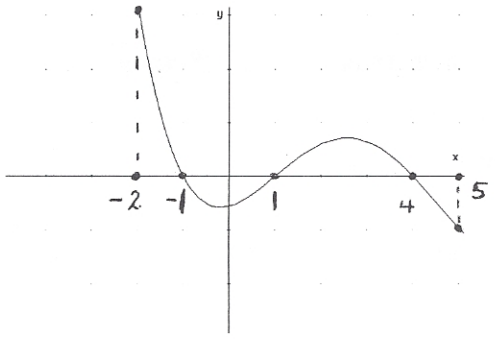
16. For $x > 0$, $\frac{d}{dx}\int_x^0 \frac{dt}{2+t^3} =$

a) $-(2+x^3)^{-1}\, t\, \frac{1}{3}$ b) $-(2+x^3)^{-1}$ c) $\frac{1}{2}x + 4x^4$ d) $(2+x^3)^{-1}$ e) $-\frac{1}{2} - 16t^3$

17. For $-1 < x \le 0$, $\frac{d}{dx}\sin^{-1}\sqrt{1-x^2} =$ a) $\frac{-x}{\left(\sqrt{1-\sqrt{1-x^2}}\ \sqrt{1-x^2}\right)}$

b) $\frac{1}{2x\sqrt{1-x^2}}$ c) $-\frac{1}{\sqrt{1-x^2}}$ d) $-\frac{1}{2x\sqrt{1-x^2}}$ e) $\frac{1}{\sqrt{1-x^2}}$

18. The following is a graph of $f'(x)$ for a function $f(x)$:



The graph of $f(x)$ is decreasing on

a) $(-2,0) \cup (3, 5)$ b) $(-2, 1)$ c) $(-1, 1) \cup (4, 5)$ d) $(-2, 1) \cup (1, 4)$ e) $[-2, 0] \cup [3, 5]$

19. Suppose $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -3$, $f'(4) = 8$, and $f'(5) = 11$. What is $F'(2)$?

a) 44 b) -12 c) 55 d) 32 e) 40

20. The $\lim_{x\to\infty} \frac{-4+\sqrt{x^2+1}}{7+\sqrt{2x^2+1}}$ is equal to

a) $-\frac{4}{7}$ b) $\frac{1}{\sqrt{2}}$ c) $\frac{4}{5}$ d) $-\frac{1}{3}$ e) $\frac{7}{10}$

21. If $\log_b x = 3$ and $\log_b y = 5$, then $\log_b\left(\frac{y^3}{b^2 x}\right) =$

a) $\frac{125}{3b}$ b) 14 c) $5b^2$ d) 10 e) 20

22. The graph of the equation $3x^2 - 18x + 5y^2 + 20y + 50 = 0$ is

a) an ellipse with center (3, -2)
b) a hyperbola with center (3, -2)
c) the empty set
d) an ellipse with center (-3, 2)
e) an ellipse with center (9, -10)

23. What are the rectangular coordinates of the point with polar coordinates $\left(-4, -\frac{5\pi}{4}\right)$

a) $(\sqrt{8}, -\sqrt{8})$ b) $\left(2\sqrt{2}, 2\sqrt{2}\right)$ c) $\left(-\sqrt{8}, \sqrt{8}\right)$ d) $\left(\sqrt{2}, -2\sqrt{2}\right)$ e) $\left(-2\sqrt{2}, -2\sqrt{2}\right)$

24. What is the largest positive integer k for which there is a polynomial $g(x)$ such that
$x^5 - 2x^3 - 2x^2 - 3x - 2 = (x+1)^k g(x)$

a) 0 b) 1 c) 2 d) 3 e) 4

25. For the set $S = \{a, b, c, d, e, f\}$ how many subsets of S contain c and contain either a or f

a) 64 b) 32 c) 16 d) 12 e) 24

26. What is the probability that a license plate number begins and ends in 5, given that the first and last digits are primes?

a) $\frac{1}{16}$ b) $\frac{1}{9}$ c) $\frac{1}{12}$ d) $\frac{1}{8}$ e) $\frac{1}{10}$

27. Assume $\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = -\frac{1}{4}$. What is $\det\begin{bmatrix} -3a_1 & -3a_2 & -3a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 - 3b_1 & c_2 - 3b_2 & c_3 - 3b_3 \end{bmatrix}$?

a) $-\frac{9}{2}$ b) $\frac{9}{2}$ c) $\frac{3}{4}$ d) $-\frac{3}{2}$ e) $\frac{3}{2}$

28. How many even 5-digit numbers can be formed from the integers 1, 2, 4, 6, 7, 8, if no integer can be used more than once?

a) 120 b) 360 c) 480 d) 96 e) 720

29. If $A = \begin{bmatrix} b & a \\ c & b \end{bmatrix}$, which one of the following conditions on *a, b, c* guarantees that det $A > 0$?

a) $a, c > 0$ and $b^2 > 4ac$

b) $a > 0$ and $2b^2 > ac$

c) $b > 0$ and $ac > 0$

d) $a > 0$ and $b^2 < ac$

e) $a, b, c > 0$

THE REMAINING PROBLEMS ARE PROBABLY BEST DONE WITH THE AID OF A GRAPHING CALCULATOR

30. The set of all real numbers x such that $|x-2|+|x-3| \leq 3$ is

a) [1, 4] b) [2, 3] c) {1, 4} d) (1, 4) e) (-∞, 4]

31. The best estimate of the largest root of the equation $x^2 = e^{.7x}$ is

a) 2.0075 b) 2.050 c) 3.8611 d) 3.8563 e) 8.6125

32. The best estimate of $\int_0^{\pi} \sin x^2 \, dx$ is

a) .6942 b) 1.6132 c) 0 d) .7727 e) -1.9027

33. $\lim_{n\to\infty} \left(1 + \frac{1}{n!}\right)^{2n} =$

a) 0 b) e c) e^2 d) 1 e) the limit does not exist

34. The best estimate of $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ is

a) 14 b) 14.15426 c) the series diverges d) 5.324145 e) 2.71828

35. Which number is the best estimate of the standard deviation (σ) of the following population ? {27.3, 35.2, 56.4, 85.5, 112.7}

a) 63.42 b) 31.83 c) 35.59 d) 56.4 e) 7.81

36. Using Newton's method to approximate the root of $x - \cos x = 0$ with the initial value $x_1 = 2.5$, the best estimate of x_4 is

a) 0.73909 b) 0.75036 c) 0.99985 d) 0.73925 e) 0.78540

37. The best estimate for the area between the graphs of $y = x^2 - 1$ and y = arctan x is

a) 1.6310 b) 32.5907 c) 1.6335 d) 31.2896 e) 3.6314

38. If $A = \begin{bmatrix} 3.85 & -6.21 & 5.42 \\ 6.50 & 2.33 & -4.21 \\ X & 0 & 1 \end{bmatrix}$ and det (A) = 0, the best estimate of X is

a) 49.3355 b) -3.6 c) -3.6503 d) 3.6503 e) -.8097

39. The best estimate of the minimum value of the function $f(x) = e^x - x^3$ on the interval [-1, 1] is

a) 0.7286 b) -0.4590 c) 0.2477 d) 0.7343 e) 0

40. The coefficient of x^{10} in the binomial expansion of $(x + 2)^{20}$ is

a) 184756 b) 1024 c) $6.8653 * 10^{14}$ d) 200 e) 189190144