FORTY-SECOND ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II
1998

Prepared by:

Mathematics Faculty
East Tennessee State University
Johnson City, TN

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, TN

Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

Dr. Hal Ramer, President, Volunteer State CC, Gallatin, Tennessee
Donnelley Printing Company, Gallatin, Tennessee
TRW Commercial Steering Division, Lebanon, Tennessee
Wright Industries, Inc., Nashville, Tennessee

ADVANCED TOPICS II 1998

1. The function

$$f(x)=\frac{x}{x^2+1}$$

is increasing on

(a) (-1,0) (b) $(-\infty, 1)$ (c) (-1,1) (d) $(-\infty, -1)$ (e) none of the above

2. Let

$$f(x)=\begin{cases}\dfrac{\sin(x-1)}{x^2-1}, & x\neq 1\\ A, & x=1\end{cases}$$

What value of A will make $f(x)$ continuous at $x=1$?

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1 (e) 2

3. At which value of x does $f(x)=x^4e^{-x}$ have a point of inflection?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

4. How many zeroes are at the end of 20!?

(a) 2 (b) 4 (c) 6 (d) 8 (e) 12

5. For $0 < x < \pi$, $\frac{d}{dx}\sin^{-1}(\cos x) =$

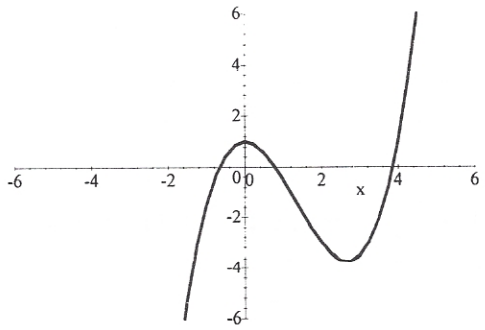
(a) -1 (b) 0 (c) 1 (d) csc(x) (e) $-\csc^2(x)$

6. If $0 \le x \le 1$, then $\frac{d}{dx}\int_x^0 \frac{dt}{3+t} =$

(a) $-\ln|3+x|+C$ (b) $\ln|3+x|$ (c) $\frac{1}{x+3}$

(d) $\ln|3+x|+C$ (e) $-\frac{1}{x+3}$

7. The following is a graph of the derivative $f'(x)$ of a function $f(x)$:



At which value of x is the graph of $f(x)$ concave upward?

(a) $x = -1$ (b) $x = 1$ (c) $x = 1$ and $x = -1$

(d) $f(x)$ always concave down (e) $f(x)$ always concave up

8. Suppose that $F(x) = f(g(x))$ and that $g(2) = 3$, $f'(3) = 4$ and $F'(2) = 3$. What is $g'(2)$?

(a) $\frac{3}{2}$ (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$ (e) undefined

9. In a world-wide chess tournament, 4096 players compete. They play one against one. The winner of each game goes to the next round. The loser goes home. How many games will be played in the tournament to determine the champion.

(a) 2048 (b) 4095 (c) 4096 (d) 8192 (e) 2^{4096}

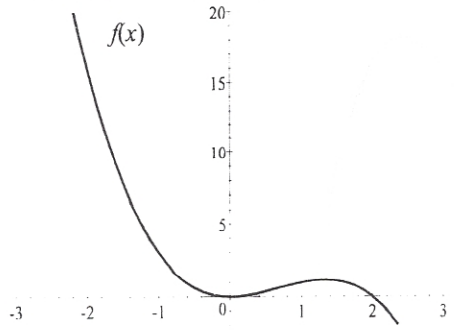
10. The limit

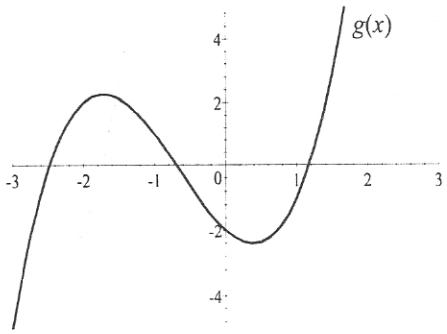
$$\lim_{x\to\infty} \frac{5+\ln(x^3)}{7+\ln(x^2)}$$

is equal to

(a) $\frac{3}{2}$ (b) $\frac{5}{7}$ (c) $\frac{7}{10}$ (d) 1 (e) ∞

11. Let $h(x) = f(g(x))$ where f and g are the functions graphed below:





Then $h'(0)$ is

(a) negative (b) zero (c) positive (d) infinite (e) undefined

12. The sum of the series

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{4n^2-1}+\ldots$$

is equal to

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) 1 (d) 2 (e) ∞

13. Acme college accepts 600 new students each year, and every year 30% of the students are lost to graduation, poor grades, transfer, etcetera. If the college continues in this way for a long period of time, how large, in numbers of students, will the college eventually become?

(a) ∞ (b) 2400 (c) 2200 (d) 2000 (e) 1800

14. If $y = Pe^{kx}$ passes through the points (1,6) and (2,4), then P is equal to

(a) 2 (b) 3 (c) 8 (d) 9 (e) 12

15. How many relative minima does the function $f(x) = 4x^2 + \cos(3x)$ have?

(a) 1 (b) 2 (c) 3 (d) 4 (e) ∞

16. If f is differentiable on $(-\infty,\infty)$, then the derivative of $\tan^{-1}\left(\frac{1}{f(x)}\right)$ is equal to

(a) the derivative of $\tan^{-1}(f(x))$

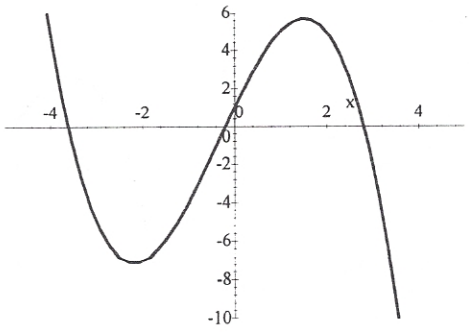
(b) the reciprocal of the derivative of $\tan^{-1}(f(x))$

(c) the square of the derivative of $\tan^{-1}(f(x))$

(d) the negative of the derivative of $\tan^{-1}(f(x))$

(e) none of the above

17. The following is the graph of the derivative, $f'(x)$, of a function $f(x)$:



If $f(0) = 2$ and $g(x) = e^x f(x)$, then $g(x)$ is

(a) positive and decreasing at $x = 0$
(b) negative and decreasing $x = 0$
(c) positive and increasing at $x = 0$
(d) negative and increasing at $x = 0$
(e) zero at $x = 0$

18. If $f(x) = x^3 - 12x^2 + 30x$, then $f(x)$ is concave upward on

(a) $(-\infty, 4)$ (b) $(-\infty, -4)$ (c) $(-4, \infty)$ (d) $(4, \infty)$ (e) (-4,4)

19. Newton's method says that if

$$x_{n+1} = x_n - 1 + 2e^{-x_n}$$

and if $x_0 = 1$, then $\lim_{n\to\infty} x_n =$

(a) 0 (b) 1 (c) 2 (d) ln(2) (e) e^2

20. Which of the following is true of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

(a) It is convergent, but not absolutely.
(b) It is divergent.
(c) It is absolutely convergent
(d) It is a p series.
(e) It is a geometric series.

21. A projectile launched vertically from the ground returns to earth T seconds later. What was the maximum height reached by the projectile? (Assume that the acceleration due to gravity is 32 ft/sec^2. Neglect the friction due to air resistance).

(a) T^2 (b) $32T$ (c) $4T^2$ (d) $16T^2$ (e) $8T^2$

22. $\int \frac{2}{1+e^{-2x}}\,dx =$

(a) $\ln\left(1+e^{-2x}\right)+C$ (b) $2\tan^{-1}\left(e^{-x}\right)+C$ (c) $2\tan^{-1}\left(e^{x}\right)+C$

(d) $\ln\left(e^{2x}+1\right)+C$ (e) $2\ln\left(e^{x}+1\right)+C$

23. $\lim_{x\to 0}\frac{\sin(1/x)}{\sin(2/x)} =$

(a) 0 (b) 2 (c) 1/2 (d) 1 (e) does not exist

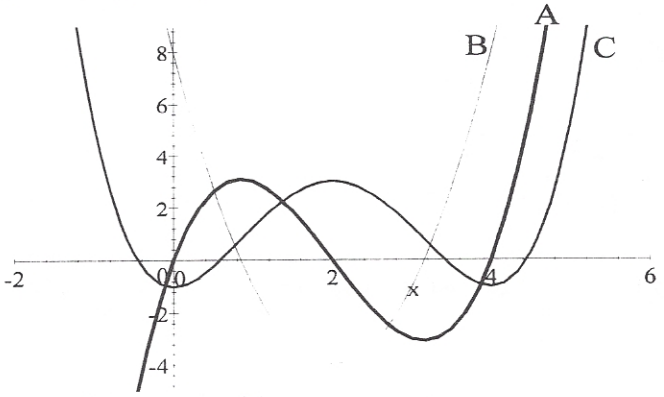
24. The function $f(x)=xe^{-x}$ is concave up on

(a) $(-\infty,2)$ (b) $(-\infty,-2)$ (c) $(-2,\infty)$ (d) $(2,\infty)$ (e) $(-\infty,\infty)$

25. Which of the following is a horizontal asymptote of $f(x)=e^{-2x}\sin\left(e^{2x}\right)$?

(a) y = -4 (b) y = -1 (c) $y=0$ (d) $y=2$ (e) $y=3$

26. The graphs of $f(x)$, $f'(x)$ and $f''(x)$ are given below. Which is which?



(a) A is $f(x)$, B is $f'(x)$, C is $f''(x)$

(b) A is $f'(x)$, B is $f''(x)$, C is $f(x)$

(c) A is $f''(x)$, B is $f'(x)$, C is $f(x)$

(d) A is $f''(x)$, B is $f(x)$, C is $f'(x)$

(e) A is $f'(x)$, B is $f(x)$, C is $f''(x)$

27. The horizontal asymptote of $f(x)=x\sin(1/x)$ is

(a) $y=0$ (b) $y=1$ (c) $y=-\pi$ (d) $y=\pi$ (e) There is no horizontal asymptote

28. If $F(x)=\int_0^x \frac{t^3+1}{t^4+1}\,dt$, then $F(x)$ is

(a) always increasing

(b) increasing on $(-\infty,-1)$ and decreasing on $(-1,\infty)$

(c) increassing on $(-\infty,-1)$ and decreasing on $(1,\infty)$

(d) decreasing on $(-\infty,-1)$ and increasing on $(-1,\infty)$

(e) always decreasing

29. Find the radius of the circle $x^2 + y^2 - 2x + 4y = 4$.

(a) $\sqrt{3}$ (b) 4 (c) 3

(d) $\sqrt{5}$ (e) 5

30. If $F\left(\frac{1-x}{1+x}\right)=x$, then

(a) $F(-2-x)=-2-F(x)$

(b) $F(-x)=F\left(\frac{1+x}{1-x}\right)$

(c) $F\left(\frac{1}{x}\right)=F(x)$

(d) $F(F(x))=-x$

(e) $F(1-x)=F(1+x)$

31. Solve for x if $2y=\frac{e^{2x}-e^{-2x}}{2}$.

a. $\frac{1}{2}\ln(2y+\sqrt{4y^2-1})$ b. $\frac{1}{2}\ln(2y-\sqrt{4y^2+1})$ c. $\frac{1}{2}\ln(2y+\sqrt{4y^2+1})$

d. $\ln(2y+\sqrt{4y^2+1})$ e. $\ln(2y-\sqrt{4y^2+1})$

32. If $T_9(x)$ is the Taylor Polynomial of order 9 of the MacLaurin Series for e^x, then $T_9'(x)=$

(a) $T_8(x)$ (b) $T_9(x)$ (c) $T_{10}(x)$ (d) e^x (e) undefined

33. The derivative of $f(x)=x^x$ is

(a) x^x (b) $x^x\ln(x)$ (c) $x^x\ln(x)+x^x$ (d) $x^x\ln(x)+1$ (e) $x^x\ln(x^x)$

34. If $f(x) = \sin(2x) + \cos(3x)$, then for which value of p below is $f(x+p) = f(x)$ for all x?

(a) $p = 2$ (b) $p = \pi$ (c) $p = 3\pi$ (d) $p = 4\pi$ (e) never

35. Let $x_{n+1} = \frac{1}{2}x_n + 1$. Then $\lim_{n\to\infty} x_n =$

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2 (e) ∞

36. $\sin\left(2\tan^{-1}(x/2)\right) =$

(a) $\frac{4x}{\sqrt{x^2+4}}$ (b) $\frac{2x}{\sqrt{2x^2+8}}$ (c) $\frac{4}{\sqrt{x^2+4}}$ (d) $\frac{2x}{2x^2+8}$ (e) $\frac{4x}{x^2+4}$

37. If $x = \cos(t)$ and $y = \sin(2t)$, then

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 4$ (c) $y^2 = 1 - 2x^2$

(d) $y = 2x\sqrt{1-x}$ (e) $4x^4 + y^2 = 4x^2$

38. Find a polar equation that has the same graph as $x^2 + y^2 = 16$.

(a) $r^2 = 16\cos 2\theta$ (b) $r = 16$ (c) $r = 4$

(d) $r^2 = 16\sec 2\theta$ (e) $r = 2$

39. The instructor of a large class has the following grading scheme: 10% of the class receives an *F*, 15% of the class receives a *D*, 30% of the class receives a *C*, 30% of the class receives a *B* and the remainder receive an *A*. A student is chosen randomly from this class. Find the probability that the student receives a *C* or better.

(a) 0.75 (b) 0.45 (c) 0.15 (d) 0.30 (e) 0.55

40. Which of the following differential equations has a solution of $y = t + e^t$?

(a) $y' + y = t$ (b) $y' + ty = 1$ (c) $y' - ty = 1$ (d) $y' - y = t$ (e) $y' - y = 1 - t$