THIRTY-NINTH ANNUAL MATHEMATICS CONTEST

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THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II   1995

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Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.
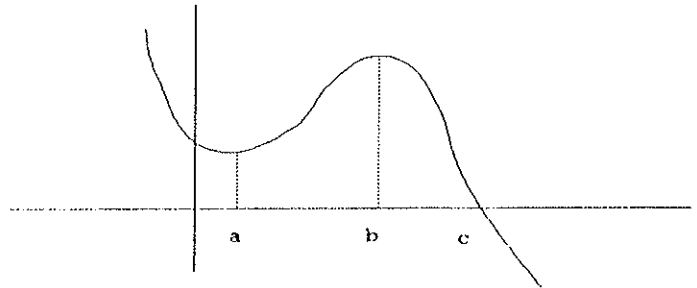
When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

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1. Below is the graph of $y = f'(x)$ where f is a differentiable function. Which of the following is true?

a) f has a local maximum at b.

b) f has a local minumum at a.

c) f has a local maximum at c.

d) f has a local minimum at c.

e) f has an inflection point at c.

2. If $\log_b x = 2$, $\log_b y = 3$, then $\log_b\left(\dfrac{bx^3}{y}\right) = ?$

a) $-\dfrac{8b}{3}$ b) 4   c) $-\dfrac{8}{3}$ d) b+3   e) 5

3. If $\det\begin{bmatrix} 2 & -3 \\ -1 & x \end{bmatrix} = 7$, then x = ?

a) 5   b) 1   c) 2   d) 0   e) -2

4. Which of the following is an antiderivative of $x^2 \cos(x^3)$?

a) $\frac{1}{3}\sin(x^3)$ d) $-\frac{1}{3}x^3 \sin(x^3)$

b) $\frac{1}{3}x^3 \sin(x^3)$ e) $-x^2 \sin(x^3) + 2x \cos(x^3)$

c) $-\frac{1}{3}\sin(x^3)$

5. If $0 \le \theta \le 2\pi$, $\cos\theta = -\dfrac{\sqrt{3}}{2}$, and $\tan\theta = \dfrac{1}{\sqrt{3}}$, then $\theta = ?$

a) $\dfrac{4\pi}{3}$ b) $\dfrac{5\pi}{6}$ c) $\dfrac{5\pi}{3}$ d) $\dfrac{11\pi}{6}$ e) $\dfrac{7\pi}{6}$

6. A rectangle has adjacent sides on the positive x and y axes and a vertex on the line $y = 1 - x/2$. What is the maximum area of such a rectangle?

a) 1   b) 2   c) 4   d) 1/2   e) $\pi/2$

7. A parabola whose directrix is the line x+y=2 and whose focus is the point (3,3) has its vertex at which point?

  a) (1,1)   b) (2,2)   c) $(2\sqrt{2}, 2\sqrt{2})$ d) $(\sqrt{3}, \sqrt{3})$ e) $(3\sqrt{2}, 3\sqrt{2})$

8. If $f'(2) = 4$ and if $g(x) = f(x^2-2)$, then $g'(2) = ?$
  a) 4   b) 8   c) f(2)   d) 16   e) 0

9. The equation $4x^2 - 16x + 9y^2 + 54y + 61 = 0$ is an equation of
a) a hyperbola with center at (2,-3).
b) an ellipse with center at (2,-3),
c) a circle with center at (2,-3).
d) an ellipse with center at (4,-9).
e) an ellipse with center at (-2,3)

10. Two marbles are drawn from a bag containing 6 red marbles and 10 white marbles. What is the probability that exactly two white marbles are drawn?
  a) 1/5   b) 3/5   c) 5/8   d) 1/8   e) 3/8

11. A spherical snowball is melting so that its volume is decreasing at a rate of 8 cubic inches per minute. At what rate is the radius decreasing when the radius is 2 inches?
  a) 8π in/min   b) $1/2\pi$ in/min   c) 16π in/min
  d) 3/2 in/min   e) $1/16\pi$ in/min

12. What are the rectangular coordinates of the point with polar

coordinates $(\frac{13\pi}{4}, -2)$?

  a) $(\sqrt{2}, \sqrt{2})$ b) $(-\sqrt{2}, -\sqrt{2})$ c) $(\sqrt{2}, -\sqrt{2})$
  d) $(-\sqrt{2}, \sqrt{2})$ e) $(-\sqrt{3}, -1)$

13. A circle of radius 1 in the first quadrant is tangent to the x-axis and the line y=x. What is the x-coordinate of the center?
  a) 2   b) 1   c) $\sqrt{2}+1$ d) $\sqrt{2}$ e) $2\sqrt{2}$

14. A function f is differentiable and concave upward on the interval (0,3). If f(1)=2 and f'(1)=1.1, which of the following is **not** a possible value of f(2)?
  a) 5   b) 10   c) 3.11   d) 3.1   e) 100

15. The function $f(x) = ax^2 + \cos(x)$ has an inflection point at the point with x-coordinate π. What is the value of a?
a) $-1/2$ b) $1/2$ c) 0 d) $\sqrt{2}/4$ e) $-\sqrt{2}/2$

16. $\lim\limits_{x \to \infty} \arctan(x)$

a) 1 b) ∞ c) $\pi/4$ d) $\pi/2$ e) π

17. $\int_0^4 \dfrac{x}{x^2 + 9}\,dx = ?$

a) $\ln(5/3)$ b) $\ln(5)$ c) $\ln(5/9)$ d) $\ln(25)$ e) $\ln(25/9)$

18. $\int_0^4 \dfrac{x}{\sqrt{x^2 + 9}}\,dx = ?$

a) $\frac{1}{2}\ln(5/3)$ b) 4 c) $\frac{1}{2}\ln(5)$ d) 2 e) $4/5$

19. $\lim\limits_{x \to 2} \dfrac{2x^2 - 8}{x - 2} = ?$

a) 4 b) 8 c) 1 d) ∞ e) $-\infty$

20. The area of the region bounded by the graphs of $y = x^2 + 5$, $y = 4$, $x = 0$, and $x = 3$ is
a) 3 b) 12 c) 4 d) $11/3$ e) $10/3$

21. If $y + \tan(xy) = 0$, then $dy/dx = ?$

a) $-\sec^2(xy)$ d) $-\dfrac{x}{\sec^2(xy)}$

b) $-\dfrac{y\sec^2(xy)}{1 + x\sec^2(xy)}$ e) $\dfrac{\sec^2(xy)}{1 + x\sec^2(xy)}$

c) $-y\sec^2(xy)$

22. If $f(x) = \int_0^x \cos(t^2)\,dt$, then $f'(x) = $?

a) $\sin(x^2)$

d) $\int_0^{x^2} \cos(t^2)\,dt$

b) $\cos(x^2)$

e) $\int_0^x \sin(t^2)\,dt$

c) $\sin(x^3/3)$

23. How many ways can a committee of 3 females and 2 males be selected from a group of 12 females and 8 males?
a) 248   b) 6160   c) 73920   d) 10944   e) 110592

24. Find all real numbers x such that $x^2 - 3x > 3 - x$.
a) All x such that x < -1 or x > 3.
b) All x such that -1 < x < 3.
c) All x such that x > 3.
d) All x such that x < -1.
e) All x such that -3 < x < 1.

25. The equation $x^4 - 5x^2 - 2x - 1 = 0$ has a solution on which of the following intervals?
a) (-2,-1)   b) (-1,0)   c) (0,1)   d) (1,2)   e) (2,3)

26. What is the y-intercept of the line which is tangent to the graph of $y = 2x^2 - x + 1$ at the point where x=1?
a) -3   b) 1   c) 0   d) -1   e) 3

27. If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and det(A) = 3, what is det(B) where

$B = \begin{bmatrix} a & 3b & c \\ 3d & 9e & 3f \\ g & 3h & i \end{bmatrix}$?

a) 3   b) 27   c) 1/9   d) 9   e) 3^6

28. What is a Cartesian equation which has the same graph as the graph of the parametric equations $x = \sin(t)-1$, $y = \cos(t)+2$?

a) $\sin(x) + \cos(y) +1 = 0$

b) $x^2 + 2x + y^2 - 4y + 4 = 0$

c) $y = \cos(\arcsin(x+3))$

d) $x^2 - 2x + y^2 + 4y + 4 = 0$

e) $x^2 + y^2 = 5$

29. What is the coefficient of x^4 in the binomial expansion of $(2x+1)^{12}$?

a) 11880   b) 190080   c) 495   d) 16   e) 7920

30. A set of exam scores is normally distributed with mean 70 and standard deviation 16. If an exam is chosen at random from this set, which of the numbers below is the best approximation to the probability that the score is greater than 86?

a) 1/3   b) 1/6   c) 1/4   d) 1/8   e) 1/9

31. $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{2x} = ?$

a) ∞ b) 0   c) 3/2   d) 2/3   e) 1

32. $\lim\limits_{h \to 0} \dfrac{\ln(2+h) - \ln(2)}{h} = ?$

a) 1   b) 0   c) 1/2   d) 2   e) ∞

33. Which of the numbers **cannot** be a root of the equation $4x^4 + bx^3 + cx^2 + dx + 6 = 0$?

a) 3/2   b) -6   c) 4   d) 1/4   e) 3/4

34. For $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & k^2-4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ k+2 \end{bmatrix}$, and $\mathbf{x}$ a 1x3 matrix, it is known

that the matrix equation $\mathbf{Ax} = \mathbf{b}$ has no solution. What is the value of k?

a) 2   b) ± 2 c) -2   d) any real number except ± 2

e) any real number except 2

35. A football coach assesses his team's chances in tomorrow's game as follows: "If it doesn't rain, we have a 70% chance of winning. If it does rain, we have only a 50-50 chance of winning. Suppose the probability of rain for the game is 0.6. What is the probability that the coach's team will win?

a) 0.58   b) 0.62   c) 0.6   d) 0.56   e) 0.54

36. Which of the following expressions is equal to $\cos^2 x \tan^2 x + 1$ for every x in the interval $(-\pi/2, \pi/2)$?

a) $\sin^2 x - \cos^2 x$

b) $\cos^2 x \sec^2 x + \sin^2 x$

c) $2 + \sin^2 x$

d) $\cos^2 x \sec^2 x$

e) $\cos^2 x$

37. A car starts from a stop and has acceleration given by the formula $a(t) = 16 - t^2$ ft/sec^2 for 4 seconds. How far does the car travel in these 4 seconds?

a) 128/3 ft   b) 248 ft   c) 448/3 ft   d) 160 ft   e) 320/3 ft

38. What is the area enclosed by one leaf of the graph with polar equation $r = \sin(3\theta)$?

a) $(\pi-3)/24$ b) $\pi/12$ c) $(\pi-3)/12$ d) 1/3   e) $\pi/6$

39. If $f(x) = \begin{cases} 1+4x-x^2 & \text{if } x \geq 0 \\ mx+b & \text{if } x < 0 \end{cases}$ and f is differentiable at 0, then

a) m=1,b=2   b) m=2,b=0   c) m=0,b=1   d) m=4,b=1   e) m=2,b=1

40. The region bounded by $y = \sqrt{x^2-1}$, y=0, x=1, x=3 is revolved about the x-axis. What is the volume of the resulting solid?

a) 8π b) 23π c) $8\pi/3$ d) $16\pi/3$ e) $20\pi/3$