THIRTY-EIGHTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II 1994

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Coordinated by D. P. Dwiggins

Scoring formula: 4R - W + 40

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish to have a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet and begin. The working time for the entire test is 80 minutes. The use of calculators is prohibited.

NOTE: 1995 Contest date, April 4

Contributors to TMTA for Annual Mathematics Contest:

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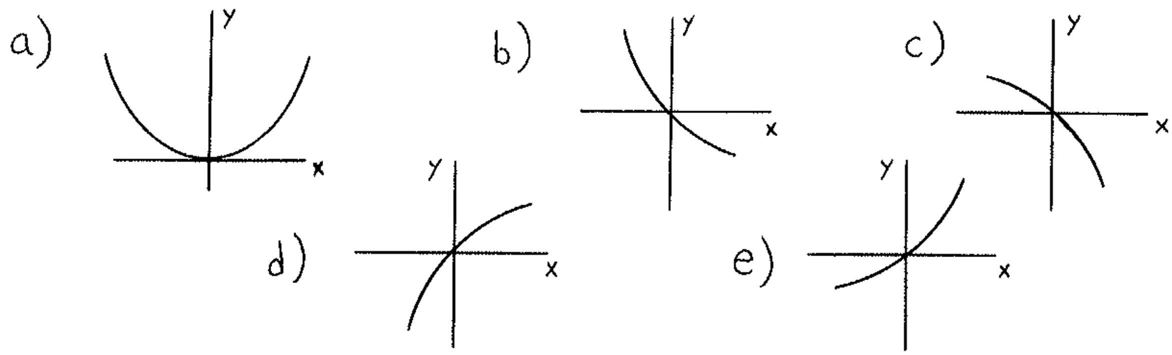
1. Which of the following depicts the graph of a function, $y = f(x)$, whose first derivative is always negative and whose second derivative is always positive?



2. If $y = 4x^2 - 4x + 4$ is defined for all real values of x, then what is the smallest possible value for y?

   a) 0   b) $\frac{1}{2}$ c) 3   d) 4   e) 6

3. Which of the following gives an **antiderivative** for $\sec x$?

   a) $\tan x$ b) $\sec x \tan x$ c) $\frac{1}{2}\sec^2 x$

   d) $\ln|\cos x|$ e) $\ln|\sec x + \tan x|$

4. Consider the graph of a hyperbola, given by the relation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad (a > 0\,,\ b > 0).$$

   Which of the following statements is **incorrect**?

   a) The center of the hyperbola is at the origin.

   b) The graph is asymptotic to lines with slope $m = \pm\, a/b$.

   c) There are no points on the graph with $|x| < b$.

   d) There are no points on the graph with $|y| < a$.

   e) The vertices of the hyperbola lie on the y-axis.

5. What is the value of $\tan\left(\dfrac{200\pi}{3}\right)$?

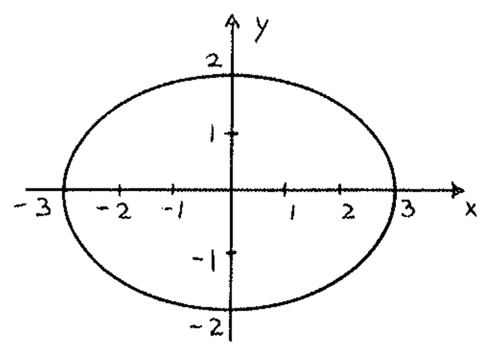
   a) $-\sqrt{3}$ b) $-\frac{1}{\sqrt{3}}$ c) $\frac{1}{\sqrt{3}}$ d) $\sqrt{3}$ e) -1

6. The volume of a sphere of radius R is given by $V = \frac{4\pi}{3}R^3$, and the corresponding surface area is given by $S = 4\pi R^2$.
Which of the following gives V as a function of S?
a) $V = \dfrac{S}{3}$ b) $V = \dfrac{1}{3}S^{3/2}$ c) $V = \left(\dfrac{S}{3}\right)3/2$
d) $V = \dfrac{S}{12\pi}$ e) $V = \dfrac{S}{6} \cdot \sqrt{\dfrac{S}{\pi}}$

7. Suppose that, in some unknown base, the following logarithmic values are given:
$\log 2 = .356$ $\log 3 = .565$ $\log 5 = .827$
What then must be the value of $\log(2.4)$?
a) .087 b) .440 c) .450 d) .486 e) .618

8. Which of the following trigonometric expressions has a constant value over the interval $0 < x < \pi/2$?
a) $\cos^2 x - \sin^2 x$ b) $\cos^{-1} x + \sin^{-1} x$ c) $\cos x + \sin x$
d) $\tan^2 x + \sec^2 x$ e) $\tan^2 x - \sec^2 x$

9. Which of the following relations describes the graph of the ellipse given at right?



a) $x^2 + y^2 = 13$
b) $(x-3)^2 + (y-2)^2 = 1$
c) $x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$
d) $x = \cos 3t,\ y = \sin 2t,\ 0 \le t \le \pi$
e) $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$

10. The function $f(x) = \dfrac{\ln x}{x}$ is defined for $x > 0$.
What is the maximum value of $y = f(x)$?

a) 0 b) 1 c) e d) $\dfrac{1}{e}$ e) $\dfrac{1}{e^2}$

11. Where does the function $f(x) = (x^2 - 5)^3$ have a point of inflection?

a) at $(0,\ -125)$ b) at $(\pm\sqrt{5},\ 0)$ c) at $(\pm 1,\ -64)$

d) at $(0,\ -125)$ and at $(\pm\sqrt{5},\ 0)$ e) at $(\pm\sqrt{5},\ 0)$ and at $(\pm 1,\ -64)$

12. It is given that $f(x)$ is continuously differentiable, with $f(0) = \pi^2$ and $f'(0) = \pi$. If $y = \sin(\sqrt{f(x)})$, then what is the value of $y' = dy/dx$ at $x = 0$?

a) $-\dfrac{1}{2}$ b) $-\dfrac{1}{2\pi}$ c) 0 d) $\dfrac{1}{2\pi}$ e) $\dfrac{1}{2}$

13. Let $f(x) = \dfrac{x^3 - x^2 + x - 1}{x^3 + x^2 - x - 1}$.

How should $f(1)$ be defined in order to make f continuous at $x = 1$?

a) Define $f(1) = 0$. b) Define $f(1) = \frac{1}{2}$.

c) Define $f(1) = 1$. d) Define $f(1) = -1$.

e) It is impossible to define $f(1)$ so that f is continuous at $x = 1$.

14. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2} = ?$

a) 0 b) $\frac{1}{2}$ c) $-\frac{1}{2}$ d) ∞ e) $-\infty$

15. $\displaystyle\lim_{x\to 0+} \left(\frac{2}{x} - \frac{1}{x^2}\right) = ?$

a) 0 b) 1 c) 2 d) ∞ e) $-\infty$

16. Given a positive integer N, let $\Delta x = \dfrac{1}{N}$, and let $x_k = \dfrac{k}{N}$ for $k = 0, 1, 2, \ldots, N$.

Using these defined values, what is the value of $\displaystyle\lim_{N\to\infty} \sum_{k=1}^{N} {x_k}^3 \Delta x$?

a) 0 b) $\dfrac{1}{4}$ c) $\dfrac{1}{3}$ d) 1 e) ∞

17. If $F(x) = \int_0^{x^2} \frac{dt}{1+t^2}$ then $\frac{dF}{dx} = ?$

a) $\frac{1}{1+x^2}$ b) $\frac{1}{1+x^4}$ c) $\frac{2x}{1+x^4}$ d) $\frac{-2x}{(1+x^2)^2}$

e) $\frac{-4x^2}{(1+x^4)^2}$

18. $\int_0^1 \frac{dx}{x^2+2x+1} = ?$

a) $\frac{1}{2}$ b) 1 c) $\frac{3}{2}$ d) $\ln 4$ e) $\frac{1}{4}\ln 4$

19. $\int_0^{\pi/2} x^2 \cos x \, dx = ?$

a) 2 b) $\pi - 2$ c) $\frac{\pi^3}{24}$ d) $\frac{\pi^2-8}{4}$ e) $\frac{\pi^2+8}{4}$

20. $\int_0^{2\pi} |\cos x| \, dx = ?$

a) 0 b) 1 c) 2 d) 3 e) 4

21. Calculate the area of the region in the xy-plane bounded by the parabola $y = x^2 - 4x$ and the line $y = 5$.

a) 26 b) 36 c) $\frac{80}{3}$ d) $\frac{110}{3}$ e) 120

22. An isosceles triangle is constructed in the xy-plane with its apex at the origin and its base vertices lying on the curve $y = \frac{16}{3} - x^2$.
Calculate the largest possible area for such a triangle.

a) $\frac{4}{3}$ b) $\frac{64}{27}$ c) $\frac{128}{27}$ d) $\frac{256}{27}$ e) $\frac{64\sqrt{3}}{9}$

23. A spherical balloon is filled with a quantity of air, for which the pressure and volume are related by the equation $PV = 1000$. (P is measured in atm = atmospheres, and V is measured in cubic units.) Suppose the volume is increasing at a constant rate given by $\frac{dV}{dt} = 50$ cubic units per minute. Calculate the rate at which the pressure is changing when the volume is equal to 200 cubic units.

a) $\frac{dP}{dt} = 20$ atmospheres per minute

b) $\frac{dP}{dt} = 1.25$ atm/min c) $\frac{dP}{dt} = 0.025$ atm/min

d) $\frac{dP}{dt} = -1.25$ atm/min e) $\frac{dP}{dt} = -0.025$ atm/min

24. A projectile is launched vertically from ground level with initial velocity $v_0 = 50$ meters per second. If the acceleration due to gravity is given by $a = -10$ meters per second squared, then what is the maximum height reached by the projectile? (Neglect the friction due to air resistance.)

a) 500 meters b) 375 meters c) 250 meters

d) 125 meters e) 62.5 meters

25. A ladder, 13 feet in length, leans against a vertical wall, with its base resting on horizontal ground. If the base of the ladder is pulled horizontally away from the wall at a constant rate of 6 feet per second, then how fast is the top of the ladder descending when the base of the ladder is 5 feet away from the wall?

a) $2\frac{1}{2}$ ft/sec b) 14.4 ft/sec c) $\frac{30}{13}$ ft/sec

d) $\sqrt{133}$ ft/sec e) $\sqrt{195}$ ft/sec

26. A triangle is constructed in the xy-plane, with its vertices given by $A = (1, 1)$, $B = (4, 5)$, and $C = (3, 3)$. Using a vector dot product, or any other method, calculate the cosine of the angle located at vertex A.

a) $\frac{7}{10}$ b) $\frac{7\sqrt{2}}{10}$ c) $\frac{-7\sqrt{2}}{10}$ d) $\frac{7\sqrt{2}}{5}$ e) $\frac{7}{100}$

27. If $\tan^{-1}(x) = \frac{1}{2}$, then which of the following gives the value of $\cos(\frac{1}{2})$?

a) $\dfrac{1}{\sqrt{1-x^2}}$ b) $\dfrac{x}{\sqrt{1-x^2}}$ c) $\dfrac{1}{\sqrt{1+x^2}}$ d) $\dfrac{x}{\sqrt{1+x^2}}$

e) $\sqrt{1+x^2}$

28. Assuming $a \cdot c \neq 0$, which of the following conditions will guarantee the polynomial $ax^3 + bx^2 + cx$ has exactly two roots?

a) $b = 0$ b) $b \neq 0$ c) $b^2 = 4ac$ d) $b^2 > 4ac$ e) $b^2 < 4ac$

29. For the polynomial $x^3 + ax^2 + bx + 12$, it is known that two of its roots are given by $x = \pm\sqrt{3}$. What must be the value of the sum $a + b$?

a) -7 b) 7 c) $4 + \sqrt{3}$ d) $4 - \sqrt{3}$

e) There is not enough information to answer the given question.

30. Which of the following gives the **best** explanation for why zero (0) does not have a multiplicative inverse?

a) Because multiplication by 0 is not well-defined.

b) Because 0 does not have an additive inverse.

c) Because 0 is not a rational number.

d) Because division by 0 is not allowed.

e) Because 0 times any number gives 0, not 1.

31. Solve the matrix equation $\mathbf{AX} = \mathbf{B}$ for $\mathbf{X}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 5 & 1 \\ 5 & 8 & 2 \end{pmatrix}.$$

a) $\mathbf{X} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ b) $\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

c) $\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ d) $\mathbf{X} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$

e) There is no solution, since **A** and **B** are not the same size.

32. Consider the equation $\mathbf{MX} = \mathbf{B}$, where $\mathbf{X}$ and $\mathbf{B}$ are column vectors of length 3, and $\mathbf{M}$ is a 3×3 matrix. Suppose that row reduction is used to solve the equation for $\mathbf{X}$, and the following augmented matrix is obtained:

$$\left(\begin{array}{ccc|c} 1 & 4 & 5 & 7 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Which of the following statements is correct?

I. $\mathbf{MX} = \mathbf{B}$ has no solution.

II. $\mathbf{MX} = \mathbf{B}$ has infinitely many solutions.

III. $\mathbf{M}$ does not have a multiplicative inverse.

a) I only b) II only c) III only

d) both I and III e) both II and III

33. Using the digits 0 through 9, how many 4-digit numerals can be made if no repeated digits are allowed and the first digit can not be 0 or 1?

a) 210 b) 1680 c) 4032 d) 5760 e) 8000

34. Each time an archer shoots at a target, she has a 20% probability of hitting the bullseye. If the archer takes 3 shots at the target, what is the probability that she will hit the bullseye exactly once?

a) 60% b) 48.8% c) 38.4% d) 12.8% e) $6\frac{2}{3}\%$

35. Suppose A and B are possible outcomes for a given event, with probabilities $P(A) = 0.25$ and $P(B) = 0.50$. If the probability that both A **and** B will occur is equal to 0.10, then what is the probability that either A **or** B will occur?

a) 0.55 b) 0.65 c) 0.75 d) 0.85 e) 0.95

36. To the nearest whole number, what is the standard deviation for the set of data values $\{25, 14, 29, 18, 14\}$?

a) 0 b) $2\frac{1}{2}$ c) 6 d) 20 e) 37

37. Suppose a game is played in which the outcome depends upon the value of X, randomly chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Also suppose the following:

If X is odd, the player wins \$1.

If $X = 2$, 4, or 6, the player wins \$2.

If $X = 8$ the player wins \$4.

If $X = 10$ the player wins \$10.

If this is to be a fair game, then how much should the player be expected to pay each time the game is played?

a) \$1.00 b) \$1.70 c) \$2.50 d) \$5.00 e) \$6.25

38. On a given day, the weatherman reported the normal high temperature to be $64°F$ and the record high temperature to be $79°F$. Assuming the daily high temperatures form a normal distribution, what should you expect the standard deviation to be?

a) $5°F$ b) $7\frac{1}{2}°F$ c) $10°F$ d) $15°F$ e) $30°F$

39. A sequence $< x_1, x_2, \ldots, x_n, \cdots >$ is defined recursively by taking $x_1 = 1$ and defining

$$x_{n+1} = 1 + \frac{2}{1 + x_n} \text{ for } n \geq 1.$$

What is the value of $x = \lim_{n \to \infty} x_n$?

a) 0 b) 1 c) $\frac{5}{3}$ d) $\sqrt{3}$ e) ∞

40. It is given that $\ln i = i\pi/2$, where $i = \sqrt{-1}$.

Given this information, what must be the value of i^i?

a) i b) $-i$ c) -1 d) $e^{-\pi/2}$ e) $\ln(\pi/2)$