THIRTY-SEVENTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish to have a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet and begin. The working time for the entire test is 80 minutes.

NOTE: 1994 CONTEST DATE--APRIL 12

1. What is the value of $\cos \frac{304\pi}{3}$?

a) $\frac{-1}{2}$ b) $\frac{1}{2}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{-\sqrt{3}}{2}$ e) $\frac{\sqrt{2}}{2}$

2. What is the value of $\arcsin(\sin \frac{3\pi}{4})$?

a) $\frac{3\pi}{4}$ b) $\frac{\sqrt{2}}{2}$ c) $\frac{-\pi}{4}$ d) $\frac{\pi}{4}$ e) $\frac{-\sqrt{2}}{2}$

3. Consider the following five quadratic equations in two variables, x and y:

I. $x^2 + 4y^2 + 6x + 5 = 0$

II. $x^2 - 4y^2 + 6x + 5 = 0$

III. $x^2 - 4y^2 + 6x + 9 = 0$

IV. $x^2 - 4y + 6x + 9 = 0$

V. $xy - 9 = 0$

For which of these equations will the graph of the equation in the xy–plane be a hyperbola?

a) Only V   b) Only IV   c) Only I   d) {II, III, V}   e) {II, V}

4. Find all real numbers x, such that $\frac{1}{x+2} + 1 < \frac{1}{x-2}$.

a) All real numbers except 2 and –2   b) All x such that $4 < x^2 < 8$
c) All x such that $-2 < x < 2$ d) All x such that $2 < x < 2\sqrt{2}$
e) All x such that $x > 2$

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5. Which of the following lines is perpendicular to the line $2x - y + 3 = 0$?

a) $2x + y + 1 = 0$ b) $2x - y - 3 = 0$ c) $x + 2y + 1 = 0$

d) $\dfrac{x}{\sqrt{2}} - \dfrac{y}{\sqrt{2}} + 2\sqrt{2} = 0$ e) $x - 2y + 5 = 0$

6. If A and B are numbers such that ln A = 2.351 and ln B = −1.042, then what is the value of $\ln\left(\dfrac{A^2}{B}\right)$?

a) 3.660   b) 5.744   c) −5.3044155   d) 5.3044155   e) −5.744

7. A function f is defined by $f(x) = \dfrac{4x - 16}{3x^2 - 12x}$ if $x \neq 4$. If f is continuous at 4, what is the value of f(4)?

a) 0   b) $\dfrac{4}{3}$ c) $\dfrac{1}{3}$ d) $\dfrac{-1}{3}$ e) undefined.

8. Find $\dfrac{dy}{dx}$ when x = 0 if $y = \sqrt{f(x)g(x)}$ and f(0) = 18, g(0) = 2, f'(0) = 3, and g'(0) = 5.

a) 8   b) 6   c) $\sqrt{15}$ d) $\dfrac{1}{12}$ e) undefined

9. What is the slope of the tangent line to the graph of $y^2 + 3xy + 7x - 11x^2 = 0$ at the point (1,1)?

a) $\dfrac{-10}{3}$ b) $\dfrac{10}{3}$ c) 4   d) $\dfrac{12}{5}$ e) 0

10. What is the value of $\log_{10} e^{-\ln .001}$?

a) 3   b) −3   c) $\dfrac{1}{3}$ d) 1000   e) .001

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11. Assume that for each integer k, $1 \le k \le n$, $x_k = \frac{k}{n}$. Let $\Delta x = \frac{1}{n}$ and let $f(x) = x^2$.

What is the value of $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(x_k) \, \Delta x$?

a) 2   b) $\frac{1}{3}$ c) 1   d) e   e) undefined

12. What is the value of the definite integral $\int_0^\pi \sin(x) \, dx$?

a) 1   b) –2   c) $\frac{\pi}{2}$ d) 2   e) $\sin(\pi)$

13. What is the value of the integral $\int_1^2 \frac{dx}{\sqrt{2x + 3}}$?

a) $2 - \sqrt{2}$ b) $\sqrt{7}$ c) $\sqrt{5}$ d) $\sqrt{5} - \sqrt{7}$ e) $\sqrt{7} - \sqrt{5}$

14. What is the value of the integral $\int_0^a \sqrt{a^2 - x^2} \, dx$?

a) a b) πa c) πa^2 d) $\frac{\pi a^2}{2}$ e) $\frac{\pi a^2}{4}$

15. What is the value of $\int_0^1 x^2 \, e^{3x} \, dx$?

a) $\frac{5e^3 - 2}{27}$ b) $\frac{e^3 - 2}{27}$ c) $\frac{e^3}{9}$ d) $\frac{5e^3}{27}$ e) $\frac{e^3}{3} - 1$

16. Find an equation for the tangent line to the curve $y = \frac{2}{x^2 + 1}$ at the point $(-1, 1)$.

a) $y = \frac{-x}{2} + \frac{1}{2}$ b) $y = x + 2$ c) $y = \frac{x}{2} + \frac{3}{2}$

d) $y = x$ e) $y = -x$

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17. If $y = (x^2+1)^{-1/2}$, find $\frac{dy}{dx}$.

a) $(x^2+1)^{-3/2}$

b) $\dfrac{1}{2\sqrt{x^2+1}}$

c) $\dfrac{x}{(x^2+1)^{3/2}}$

d) $\dfrac{-1}{2(x^2+1)^{3/2}}$

e) $\dfrac{-x}{(x^2+1)^{3/2}}$

18. If $y = \sin 2x$, find $\frac{d^2y}{dx^2}$.

a) 2 cox 2x

b) $-4 \sin 2x$

c) $-8 \sin x$

d) 4sin 2x

e) $-2\sin 2x$

19. Let A denote the area beneath the graph of $y = e^{x^2}$ on the domain $0 \le x \le t$. What is the value of $\frac{dA}{dt}$ when $t = 3$?

a) $6e^9$ b) e^9 c) e^6 d) e   e) undefined

20. Which of the following is true for all $x \ge 0$?

a) $\sin x \le x$

b) $\sin x < x$

c) $\sin x < \cos x$

d) $\sin x > x$

e) $\sin x \ge x$

21. For all A and B, what is the value of the expression

$$\frac{\sin A \cos B + \cos A \sin B}{\cos^2A \cos^2B + \cos^2A \sin^2B + \sin^2A \cos^2B + \sin^2A \sin^2B} \ ?$$

a) $\sin(A + B)$

b) $\sin(A - B)$

c) $\frac{1}{2}\sin(A + B)$

d) $\tan(A + B)$

e) $\cos(A + B)$

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22. Let each of a_i, b_i, c_i ($i = 1, 2, 3$) be a real number and suppose the system of equations:

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0.$$

has a solution (x, y, z), where $xyz \neq 0$. Pick a correct answer from the following:

a) There is exactly one such solution to the system.
b) There are exactly three such solutions to the system.
c) There are infinitely many solutions to the system.
d) The matrix $M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ has a determinant equal to 1.
e) The matrix M has an inverse.

23. Solve the following matrix equation for X:

$$\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

a) $X = \begin{bmatrix} 1 & -1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$
b) $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
c) $X = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ 3 & \frac{4}{3} \end{bmatrix}$

d) $X = \begin{bmatrix} -2 & -2 \\ \frac{5}{3} & 2 \end{bmatrix}$
e) $X = \begin{bmatrix} \frac{1}{3} & \frac{-1}{3} \\ \frac{5}{3} & \frac{-1}{3} \end{bmatrix}$

24. What is the sum of all the roots of the equation $x^4 + 3x^3 + 2x^2 + 5x + 11 = 0$?

a) −3   b) 11   c) 3   d) −5   e) 5

25. Find the value of the sum, $\sum_{k=1}^{100} i^k$, where i is the complex number $\sqrt{-1}$.

a) 1   b) 0   c) $\frac{1}{i-1}$ d) −1   e) 100

26. Suppose that f is a quadratic function   (i.e., $y = f(x) = ax^2 + bx + c$ for some constants a,b,c) that gives rise to the following table:

x	y	y'	y"
1	2	−1	−1
0	f(0)		

What is the value f(0)?

a) 4   b) $\frac{7}{2}$ c) $\frac{5}{2}$ .. d) 1   e) $\frac{1}{2}$

27. A particle moving in a straight line has acceleration $a = 12t - 6$ for $t \geq 0$.
When $t = 0$ the particle is located at $s_0 = 1$ and has velocity $v_0 = -1$. A formula for the position of the particle at any nonnegative time t is given by:

a) $12t^3 - 6t^2 - 1$ b) $2t^3 - 3t^2 - t + 1$ c) $2t^3 - 3t^2 + t - 1$
d) $12t^3 - 6t^2 + 1$ e) $2t^3 - t^2 - t + 1$

28. The notation $\sum\limits_{k=1}^{5} (-1)^{k+1}(2k + 1)^2$ stands for which of the following expressions?

a) $2^2 - 4^2 + 6^2 - 8^2 + 10^2$ b) $3^2 - 5^2 + 7^2 - 9^2 + 11^2$
c) $1 - 2^2 + 3^2 - 4^2 + 5^2$ d) $-3^2 + 5^2 - 7^2 + 9^2 - 11^2$
e) $2^2 * 4^2 * 6^2 * 8^2 * 10^2$

29. Which of the following numbers represents the area which is inside the parabola $y = x^2 - 4$ and below the line $y - 2x + 1 = 0$?

a) 9   b) $\frac{4}{3}$ c) $\frac{32}{3}$ d) $\frac{16}{3}$ e) $\frac{8}{3}$

30. The first quadrant area under the curve $y = 2x - x^2$ is revolved about the

x–axis. Which of the following represents the volume of the resulting figure?

a) $\frac{4}{3}$ b) 2π c) 3   d) $\frac{16\pi}{15}$ e) $-\frac{16\pi}{15}$

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31. Which of the following is the graph of the polar equation $r = 4 \sin\theta$?

a) circle   b) 4 leaf rose   c) cardioid
d) spiral   e) a "figure eight"

32. Each time he bats, the probability that Mighty Casey hits a home run is 0.3. Every time that Mighty Casey bats, he either hits a home run or strikes out. In the last game of the championship series, Mighty Casey comes to bat four times. What is the probability that he hits at least one home run in this game?

a) $4(0.3)$ b) $(0.3)^4$ c) $4(0.3)(0.7)^3$ d) $(0.7)^4 + (0.3)^4$ e) $1 - (0.7)^4$

33. A club has 20 members. Of the twenty members 8 are boys and 12 are girls. A five member commitee is to be selected; the committee will have three girls and two boys. In how many ways can the committee be selected?

a) 10944   b) 248   c) 110592   d) 6160   e) 73920

34. The mean (average) salary at the XYZ corporation is $36,000 with a standard deviation of $6,500. The president of the XYZ corporation proposes that each employee receive an increase in salary of $1800. This is called the "flat raise proposal." The personnel manager proposes that each employee should receive an increase in salary of 5%. This is called the "percentage raise proposal." It is true that for each of the two proposals, the mean salary would rise to $37,800. Which of the following statements is true?

a) Under both proposals, the standard deviations would be increased by the same amount.

b) Under both proposals, the standard deviations would not be changed.

c) Under the flat raise proposal, the standard deviation would increase; but under the percentage raise proposal, the standard deviation would not be changed.

d) Under the flat raise proposal, the standard deviation not be changed; but under the percentage raise proposal, the standard deviation would increase.

e) Under both proposals, the standard deviations would be increased; but the increase would be larger under the percentage raise proposal.

35. The polynomial equation $x^4 + x^3 + x^2 + x + 1 = 0$ has exactly how many rational roots?

a) 4   b) 3   c) 2   d) 1   e) 0

36. Let $f(x) = x^5 + ax^2 + bx + c$, where a, b, and c are real numbers. Suppose that $1 + 3i$ and $2 + 5i$ are roots of $f(x) = 0$. At how many points does the graph of f intersect the x–axis?

a) 3   b) 5   c) 1   d) 0   e) Infinitely many

37. Consider the two vectors in 3–dimensional space:

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \text{ and } \mathbf{w} = \mathbf{i} + \mathbf{j} + c\mathbf{k}, \text{ where c is a scalar.}$$

For which value of c will $\mathbf{w}$ be perpendicular to $\mathbf{v}$?

a) 5   b) –1   c) 1   d) 6/5   e) 0

38. If r is the radius of a sphere, then its volume is $\frac{4}{3}\pi r^3$. A spherical balloon is being blown up at the rate of 4 cubic feet/second. How fast is the radius growing when the radius is 3 feet?

a) $\frac{1}{9\pi}$ ft/sec   b) $\frac{1}{36\pi}$ ft/sec   c) $\frac{\pi}{9}$ ft/sec

d) $\frac{\pi}{4}$ ft/sec   e) 9π ft/sec

39. A farmer will erect fencing for a large rectangular pasture by fencing in one large rectangle. Because he has two animals which are incompatible, he then divides the pasture into two rectangular areas by dividing it with a fence running from one side of the rectangle to the opposite side. If he has a total of 1200 yards of fence to use, what is the maximum area of pasture he can enclose?

a) 30,000 sq. yd.   b) 60,000 sq. yd.   c) 20,000 sq. yd.
d) 6,000 sq. yd.   e) 5,000 sq. yd.

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40. Newton's Method for finding $\sqrt{3}$ simplifies to $x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$. Given that $x_0 = 2$ what is x_2?

a) 2　　　　b) 1　　　　c) 1.7　　　　d) $\frac{58}{26}$　　　　e) $\frac{97}{56}$