THIRTY-SITXH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics II   1992

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Scoring formula: 4R - W + 40

Edited by: Larry Bouldin, Roane State Community College, Harriman, TN

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish to have a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet and begin. The working time for the entire test is 80 minutes.

Contributors to TMTA for Annual Mathematics Contest:

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NOTE: 1993 CONTEST DATE--APRIL 6

1. Find the corresponding rectangular equation by eliminating the parameter:
$x = 3 + 2 \cos \theta, \quad y = 1 + \sin \theta$.

a) $x^2 + 4 y^2 - 6 x - 8 y + 9 = 0$ b) $x^2 - 4 y^2 - 6 x + 8 y + 1 = 0$

c) $\dfrac{x^2 - 9}{4} + \dfrac{y^2 - 1}{4} = 1$ d) $x = 2y + 1$ e) $(x - 3)^2 + (y - 1)^2 = 5$

2. The value of $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)^2 \, dx$ is

a) $\pi/2$ b) $\pi/2 - 1$ c) $\pi/2 + 1$ d) 0 e) π

3. If i denotes the imaginary unit, then the exact value of $\displaystyle 1 + \sum_{n=1}^{10} i^n$ is

a) 1 b) $1 + i$ c) i d) 0 e) -1

4. What is the binary (base-2) representation of the base-10 numeral 99?

a) 110011 b) 111111 c) 100011 d) 1100011 e) 101101

5. How many subsets does the set $\{q,w,e,r,t,y\}$ have? (Include the empty set in your count.)

a) 1000000 b) 12 c) 256 d) 1024 e) 64

6. At noon a vessel is sailing due north at the uniform rate of 15 mph. Another vessel, 30 mile due north of the first vessel, is sailing due east at the uniform rate of 20 mph. At what rate is the distance between the vessels changing at the end of 1 hour?

a) 20 mph b) 11 mph c) 25 mph d) 7 mph e) 10 mph

7. Solve for the 2 x 2 matrix **X**:
$$X \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$$

a) $\begin{bmatrix} -4 & -2 \\ -14 & 5 \end{bmatrix}$ b) $\begin{bmatrix} -4 & -2 \\ -14 & -5 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 1 \\ -14 & -11 \end{bmatrix}$ d) $\begin{bmatrix} 2 & -1 \\ -14 & -11 \end{bmatrix}$ e) $\begin{bmatrix} 4 & -2 \\ -14 & -5 \end{bmatrix}$

8. Let **p** be the statement: $\sin (\sin^{-1} \pi/3) = \pi/3$
and **q** be the statement: $\sin^{-1} (\sin \pi/3) = \pi/3$
Then which of the following holds:

a) **p** is TRUE; **q** is FALSE b) **p** is FALSE, **q** is TRUE c) Both **p** and **q** are TRUE
d) Both **p** and **q** are FALSE e) $p \Rightarrow q$

9. An urn contains 8 red balls and 4 white balls. Two balls are drawn without replacement. What is the probability of drawing at least 1 red?

a) $\frac{16}{33}$ b) $\frac{8}{9}$ c) $\frac{14}{33}$ d) $\frac{4}{9}$ e) $\frac{10}{11}$

10. The resistance R of two resistors with resistances R_1 and R_2 when connected in parallel is given

$$R = \frac{R_1 R_2}{R_1 + R_2}.$$

If the resistance, R, of the two resistors is 3.4 Ohms and one of the resistors has resistance $R_1 = 4.6$ Ohms, what is the approximate resistance of the other resistor?

a) 1.2   b) 8   c) 15.64   d) 3.4   e) 13.033

11. $\lim\limits_{x \to 0^+} \frac{|x|}{x} =$

a) $-\infty$ b) -1   c) 1   d) ∞ e) the limit does not exist

12. Find $\frac{\log_7 81}{\log_7 3}$

a) 27   b) 4   c) 21   d) 98   e) 567

13. What is the probability that a telephone number ends in 7, given that the last digit is a prime number?

a) $\frac{1}{10}$ b) $\frac{1}{4}$ c) $\frac{1}{5}$ d) $\frac{2}{5}$ e) $\frac{1}{7}$

14. If $(2x-1)^{24}$ is expanded by the binomial theorem, then the coefficient of x^2 is given by

a) 4   b) 1104   c) -1104   d) 552   e) -552

15. Given the graph of the function f below, which one of the graphs a) - e) is most likely to be the graph of the derivative of f?
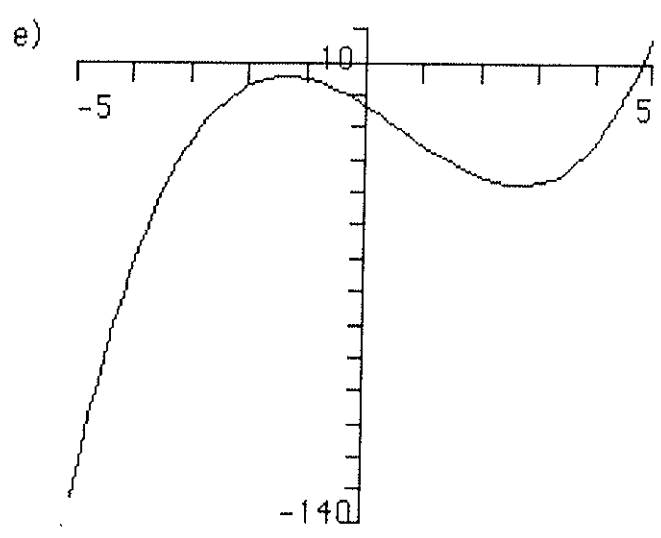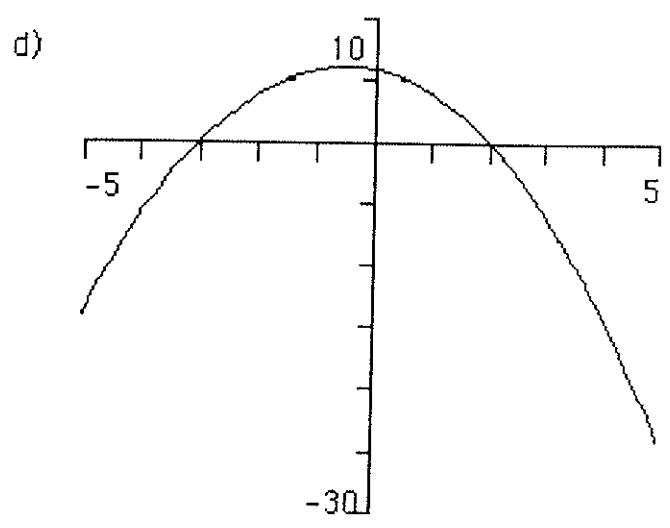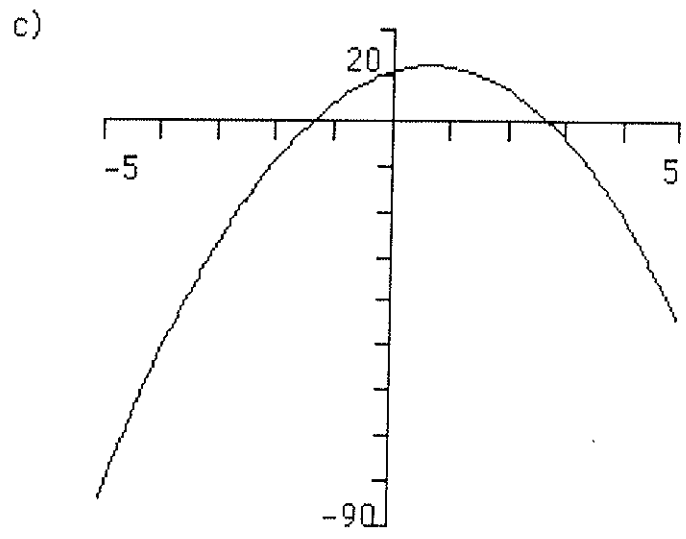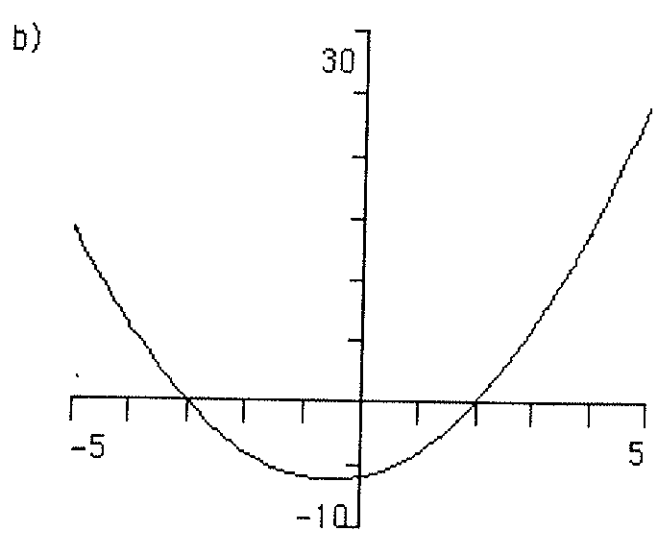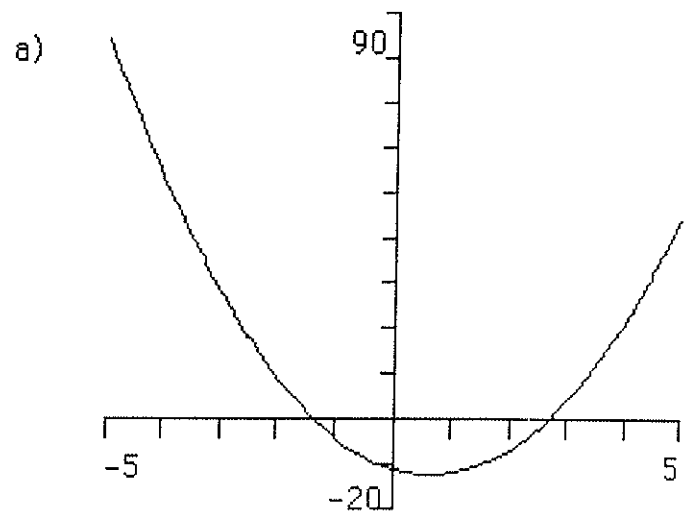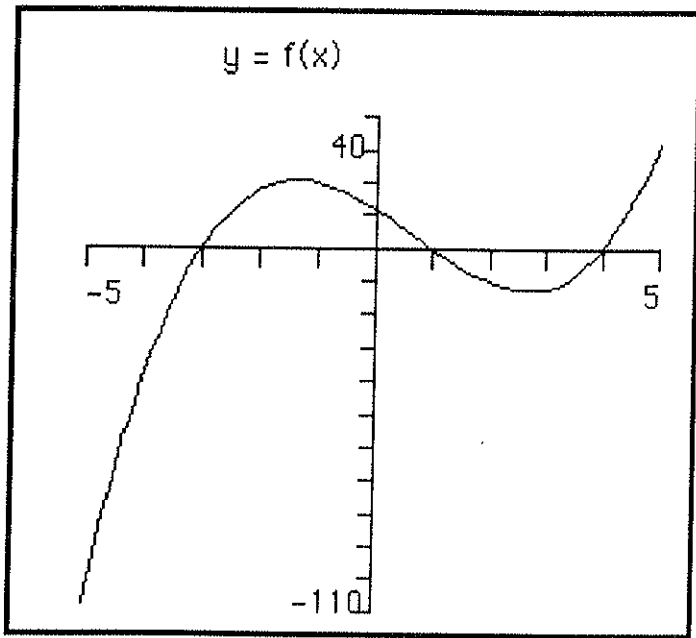
16. Let $\exp(x) = e^x$. Compute $\lim_{x \to 0} \exp\left[\dfrac{\sin(x^2)}{x^2}\right]$

    a) -1   b) e^{-1} c) e d) 1   e) 0

17. Assume $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -\dfrac{1}{3}$. What is $\det\begin{bmatrix} -2a & -2b & -2c \\ 3g & 3h & 3i \\ d-2g & e-2h & f-2i \end{bmatrix}$?

    a) 18   b) 2   c) -2   d) 9   e) -9

18. Find the equation of the curve for which $y'' = 2x$, if the line $2x - y - 3 = 0$ is tangent to the curve at the point $(1,-1)$.

    a) $y = x^2 + 1$ b) $y = \dfrac{1}{3}x^3 - 2x + \dfrac{2}{3}$ c) $y = \dfrac{1}{3}x^3 - 3x + \dfrac{5}{3}$ d) $y = \dfrac{1}{3}x^3 + x - \dfrac{7}{3}$ e) $y = x^2 - 2$

19. $\lim_{x \to 1^+} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right) = $ _____

    a) -1   b) 1   c) $-\dfrac{1}{2}$ d) $\dfrac{1}{2}$ e) $\dfrac{2}{e^2}$

20. $\displaystyle\int \dfrac{dx}{1 + e^x} = $ _____

    a) $\ln|1 + e^x| + C$ b) $-\ln|1 + e^x| + C$ c) $\tan^{-1} e^x + C$

    d) $-\ln|1 + e^{-x}| + C$ e) $\ln|1 + e^{-x}| + C$

21. A license tag has 4 digits. If the first digit cannot be zero, and no digit can be used more than once, how many different tags can be made?

    a) 3024   b) 9000   c) 4536   d) 6480   e) 6561

22. The inflection point(s) of $f(x) = xe^{-2x}$ are at $(x,y) = $
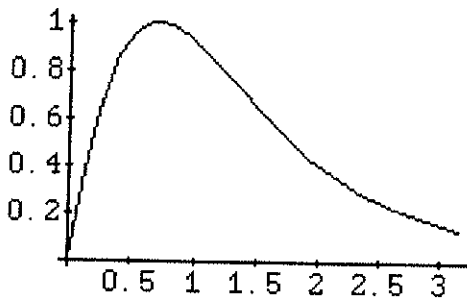
    a) $(1, e^{-2})$ b) $(1, e^{-2})$ and $(0, 0)$ c) $(0, 0)$

    d) $(\frac{1}{2}, \frac{1}{2} e^{-1})$ e) $(1, 0)$

23. Consider the line segments joining $(2, 1)$ and $(-2, 3)$ to the origin. The cosine of the angle between these line segments is given by
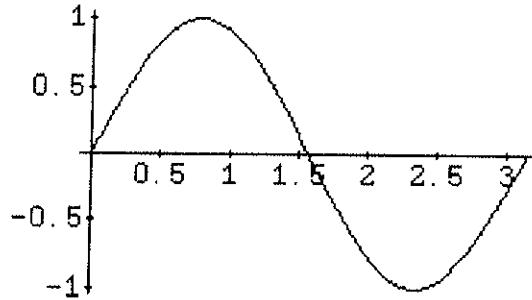
    a) 0   b) 1   c) $\dfrac{\sqrt{65}}{65}$ d) $\dfrac{-1}{\sqrt{65}}$ e) $\dfrac{-1}{8}$

24. Which one of the following graphs represents a part of the graph of $f(x) = \sin(\pi e^{-x})$.
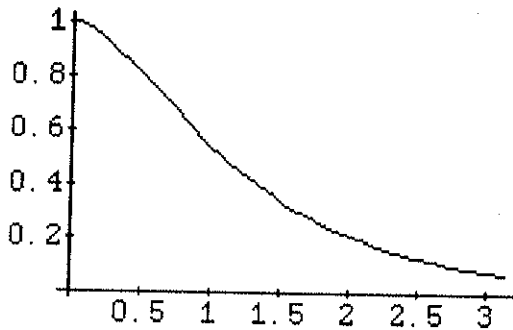
a)

b)

c)

d)
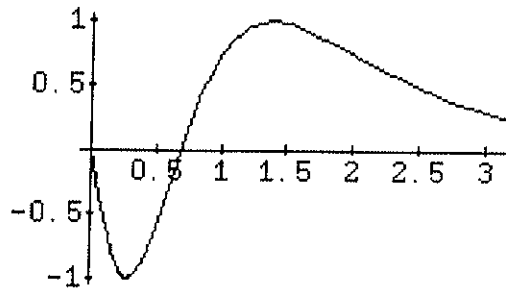
e)
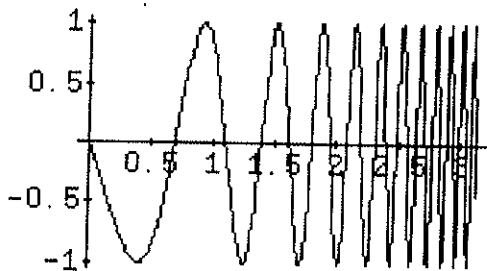
25. The value of $\displaystyle\int_0^1 e^{(x^2)} dx$ is closest to

a) -1   b) 0   c) 1   d) 2   e) 3

26. $\displaystyle\lim_{x \to 1} \frac{1 - x^3}{2 - \sqrt{x^2 + 3}}$

a) 4   b) 6   c) 1   d) 0   e) $\dfrac{1}{2}$

27. A car manufacturer has two suppliers of batteries. The quality control department tells him that 2% of the batteries from supplier A are defective and 3.5% of the batteries from supplier B are defective. The batteries from B cost the manufacturer less but have a higher percentage of defective batteries. If the manufacturer wants to have 2500 batteries with 2.6% defective, how many batteries should he buy from supplier A?

a) 500   b) 1500   c) 65   d) 2250   e) 2500

28. Match the graph shown with the correct function
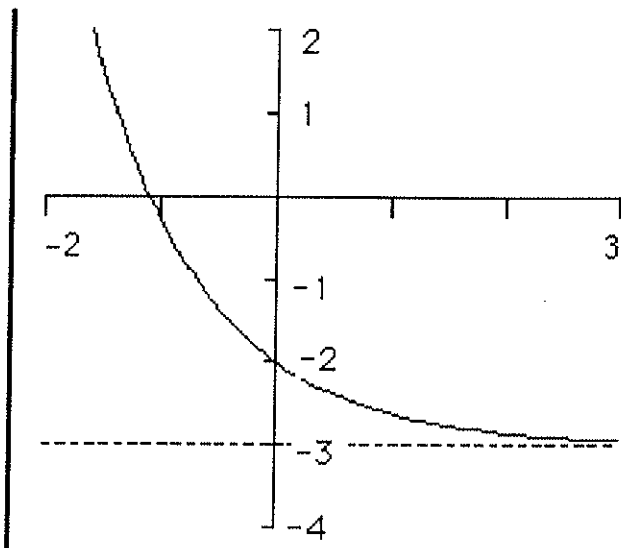
a) $f(x) = e^{x+3}$ b) $f(x) = e^{-x} - 3$ c) $f(x) = e^{x} - 3$ d) $f(x) = e^{-(x+3)}$ e) $f(x) = e^{x-3}$

29. Let c be a constant so that f(x) will be continous on the entire real line, where

$$f(x) = \begin{cases} x^3 + 2, & \text{for } x \le 3 \\ \dfrac{c}{x} - 6, & \text{for } x > 3 \end{cases}$$

Then c =

a) -3   b) 0   c) 3   d) 29   e) 105

30. What is the largest positive integer k for which there is a polynomial q(x) such that
$x^5 - x^3 + 2x^2 - 6x + 4 = (x-1)^k q(x)$?

a) 1   b) 2   c) 3   d) 4   e) 5

31. A circle is divided into 4 equal sections colored red, green, blue, and yellow. If a spinner lands on red, the player receives ten dollars; if it lands on green, the player receives twenty dollars; and if it lands on blue the player receives thirty dollars. However if the spinner lands on yellow, the player pays forty dollars. How much should the player be willing to pay to play?

a) $5.00   b) $10.00   c) $15.00   d) $20.00   e) $25.00

32. $\lim\limits_{x \to 5} \dfrac{x^2 - 3x - 10}{x - 5} =$

a) 0   b) 1   c) 5   d) 7   e) The limit does not exist

33. Which of the following is an equation of the line normal to $y = e^x$ at $(1, e)$?

a) $y = e + e^{-1} - e^{-1}x$ b) $y = e - e^{-1} + e^{-1}x$ c) $y = ex$ d) $y = 2e - ex$ e) $y = 2e + ex$

34. Which of the following numerals represent the decimal numeral 15?
   I) 1111 (base 2)   II) 120 (base 3)   III) 30 (base 5)   IV) 10 (base 15)

a) I and II only   b) I and III only   c) II and III only   d) I, II and III   e) I, II, III and IV

35. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ which one of the following conditions on a,b,c guarantees that det A > 0?

a) a, b, c > 0   b) $b^2 - 4ac \geq 0$ c) a > 0 and $b^2 < ac$
d) $ac \geq 0$ e) a,b,c > 0 and $b^2 > ac$

36. If $y = (2x + 1)^x$ for $x > -\dfrac{1}{2}$ then $\dfrac{dy}{dx} =$

a) $2x (2x+1)^{x-1}$ b) $2 (2x+1)^{x-1}$ c) is not defined for all $x > -\dfrac{1}{2}$

d) $(2x+1)^x \left(\dfrac{2x}{2x+1} + \ln (2x+1) \right)$ e) $(2x +1)^x \ln (2x+1)$

37. $\lim\limits_{x \to \frac{3\pi}{2}} \tan x$

a) $-\infty$ b) -1   c) 1   d) ∞ e) does not exist

38. Which one of the following straight lines intersects the curve $y = x^3 - x$ exactly twice?

a) $y = -2$ b) $y = 2x - 2$ c) $y = -x - 2$ d) $x = 0$ e) $y = 3x - 2$

39. If tan x = 1/2, then cos 2x =

a) $\dfrac{-4}{\sqrt{5}}$ b) $\dfrac{3}{5}$ c) $\dfrac{4}{5}$ d) $\dfrac{4}{\sqrt{5}}$ e) $\dfrac{-3}{5}$

40. Scores on a test are known to follow the normal distribution. If the mean is 450 with a standard deviation of 75, approximately what percentage of the scores is expected to be greater than 600?

a) 1%   b) 2.5%   c) 5%   d) 16%   e) 34%