THIRTY-SECOND ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

ADVANCED TOPICS II 1988

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Scoring formula: 4R - W + 40

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem there are listed 5 possible answers. You are to work each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet to page 2 and begin. When you have finished one page, go on to the next. The working time for the entire test is 80 minutes.

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1. What is the slope m of the line tangent to the curve $x^2 + 2y^2 + 4x - 8y = 24$ at the point where the curve crosses the positive y-axis?

a) m = -2/3

b) m = -1/4

c) m = -3/2

d) slope undefined

e) 1/2

2. If $\frac{d^n f(x)}{dx^n}$ denotes the n^{th} derivative of $f(x)$ what is $\frac{d^{(1000)} sin(x)}{dx^{(1000)}}$ when $x = \pi/2$?

a) 0

b) 1

c) -1

d) π

e) 1/2

3. $\lim_{x \to 3} \frac{\sin(x-3)}{x^2-9} =$

a) 1/6

b) Does not exist

c) 1

d) 0

e) 1/2

4. Given $F'(a)$ exists, which of the following are true?

1) $F'(a) = \lim_{h\to a} \frac{F(h)-F(a)}{h-a}$
2) $F'(a) = \lim_{h\to 0} \frac{F(a)-F(a-h)}{h}$
3) $F'(a) = \lim_{h\to 0} \frac{F(a+2h)-F(a)}{h}$
4) $F'(a) = \lim_{h\to 0} \frac{F(a+2H)-F(a+h)}{2h}$
5) $F'(a) = \lim_{h\to a} \frac{F(a+h)-F(a)}{h-a}$

a) Only (1) is TRUE

b) Only (1) and (2) are TRUE

c) Only (1) and (4) are TRUE

d) All are TRUE

e) Only (2) and (3) are TRUE

5. An isosceles triangle is inscribed in a circle of radius r. If the apex angle, θ, is restricted by $0 \leq \theta \leq \pi/2$, the largest value of the perimeter is:

a) $3r$

b) $3\sqrt{3}r$

c) $2(1+\sqrt{2})r$

d) $4r$

e) $\sqrt{2}(1+\sqrt{3})r$

6. $1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1) =$

a) $(2n)!/n$

b) $(2n)!/(2^n n!)$

c) $(2n)!/2^n$

d) $\frac{(2n-1)!}{2n}$

e) $(2n-1)!$

7. A combination of three balls is picked at random from a box containing five red, four white, and three blue balls. In how many ways can the set chosen contain at least one white ball?

a) 164

b) 284

c) 112

d) 160

e) 192

8. Given that

$$e = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}, \quad \text{find the sum}$$

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

a) $1 + e$

b) $1 - e$

c) $e - 1$

d) 1

e) e

9. The maximum value of $y = \frac{ln(x)}{x}$ is

a) 1

b) e

c) e^{-1}

d) $e^{1/e}$

e) 0

10. If $t > 0$, $\int_1^t ln(x^2)dx =$

a) $2[tln(t) - t + 1]$

b) $tln(t^2) - 2t$

c) $\frac{2}{t} - 1$

d) $\frac{ln(t^3)}{3}$

e) $2ln(t)$

11. The radius, r, of a right circular cone is increasing at a rate of 6 cm/min while its height h is decreasing at a rate of 15 cm/min. At the instant when $r = 50$ cm and $h = 50$ cm, the volume is:

a) Increasing at a rate of 2500 $\pi cm^3/min$.

b) Decreasing at a rate of 2500 $\pi cm^3/min$.

c) Decreasing at a rate of 7500 $\pi cm^3/min$.

d) Decreasing at a rate of 7500 $\pi cm^3/min$.

e) Unchanging.

12.

$$\text{For} \quad c > 0, f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > c \\ a + bx^2 & \text{if } |x| \leq c \end{cases}$$

If $f'(c)$ exists then:

a) $a = c$ and $b = \frac{-1}{c}$

b) $a = \frac{2}{c}$ and $b = \frac{-1}{c^3}$

c) $a = \frac{2}{3c}$ and $b = \frac{-1}{3c^3}$

d) $a = \frac{3}{2c}$ and $b = \frac{-1}{2c^3}$

e) a and b cannot be uniquely determined

13. $\frac{sin^2 2t}{(1+cos2t)^2} + 1 =$

a) $\sec^2 t$

b) $2\csc^2 t$

c) $2\tan^2 2t$

d) $1+\cos 2t$

e) $2(1-\cos 2t)$

14. $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) =$

a) $\frac{2\pi}{4}$

b) $\frac{3\pi}{4}$

c) $\frac{4\pi}{4}$

d) $\frac{5\pi}{4}$

e) $\frac{6\pi}{4}$

15. In the figure at the right assume that circle A is immovable and that circle B can roll around the circumference of A. If the radius of circle B is half that of A and it makes one trip around circle A returning to its original position, how many rotations will it make?

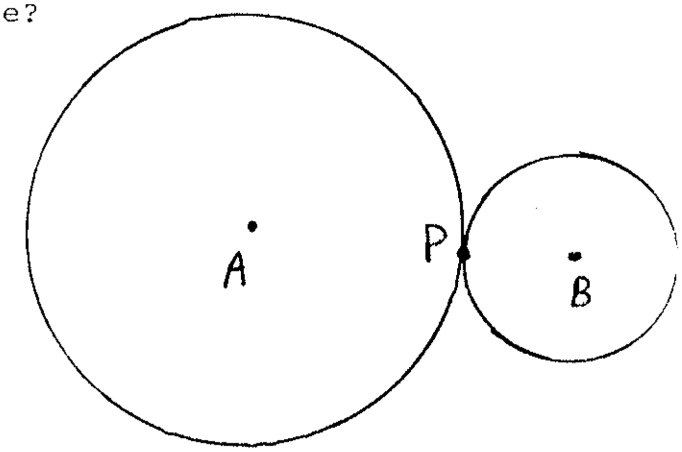
(a) 1

(b) 2

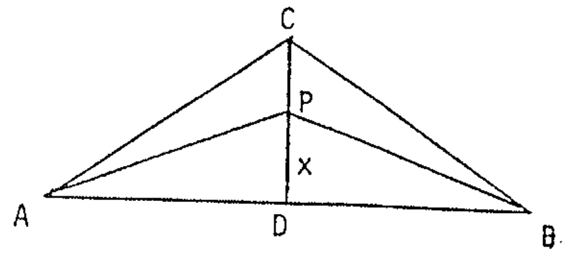
(c) 3

(d) 4

(e) $\frac{3\pi\sqrt{2}}{2}$



16. Consider the isosceles triangle ABC (see figure below) in which base $AB = 12$ and altitude $CD = 3$. Find a point P, if it exists, on segment CD such that $S = PC + PA + PB$ is a minimum.



a) $PD = 3 - 2\sqrt{3}$

b) $PD = 2\sqrt{3} - 3$

c) $P = C$

d) There is no point on segment CD satisfying condition.

e) $P = D$

17. Let S be the set of all integers satisfying $\left(\frac{n+1}{2}\right)^n > n!$ Then

a) $S = \phi$

b) $S = \{2, 3, 4, 5, 6, 7, 8, 9\}$

c) $S = \{\text{all non-negative integers}\}$

d) $S = \{\text{all integers greater than } 1\}$

e) $S = \{\text{all positive odd integers}\}$

18. If two cards are drawn at random from a standard deck of playing cards, what is the probability that either both are red or both are Jacks?

a) ${}_{26}C_2/{}_{52}C_2$

b) $({}_{26}C_2 + {}_4C_2 - 1)/{}_{52}C_2$

c) $({}_{26}C_2 +_4 C_2)/{}_{52}C_2$

d) $({}_{26}C_2 +_5 C_2)/{}_{52}C_2$

e) $1 - ({}_{26}C_2 +_4 C_2/{}_{52}C_2$

19. Consider an experiment in which two cards are drawn at random from a deck of ten cards, six of which are red and four blue. The first card is not returned to the deck before the second is taken. Led E_1 and E_2 be the events that a red and blue are obtained on the first and second draw, respectively.

Then: $P(E_1 \cap E_2) =$

a) 6/25

b) 4/15

c) 2/5

d) 3/5

e) 11/15

20. Find the polynomial $f(x)$ of the fourth degree such that

$$f(0) = f(1) = 1, \;\; f'(0) = f''(0), \;\; f'''(1) = 54$$

a) $4x^4 - 4x^3 + 1$

b) $-x^4 + x^3 + 1$

c) $5x^4 - 5x^3 + 1$

d) $3x^4 - 3x^2 + 1$

e) $3x^4 - 3x^3 + 1$

21. $\lim_{x \to 0+} (\cos x)^{\frac{1}{\tan x}} =$

a) 1

b) 0

c) ∞

d) -1

e) $\frac{\pi}{2}$

22. $\int_{-2}^{7} (x+1)^{-\frac{7}{3}} dx =$

a) Does not exist

b) $\frac{15}{16}$

c) $\frac{-15}{16}$

d) $\frac{17}{16}$

e) $\frac{-17}{16}$

23. A particle moves along the x-axis is such a way that its position at any time t is given by $x = -t^3 + 9t^2 + 25t + 29$. What is the position of the particle when the velocity is a maximum?

a) 100

b) 223

c) 158

d) 176

e) 198

24. Which of the following lines lie in the plane $2x - 5y - z = 10$

a) $\frac{x-5}{2} = \frac{y}{-5} = \frac{z}{-1}$

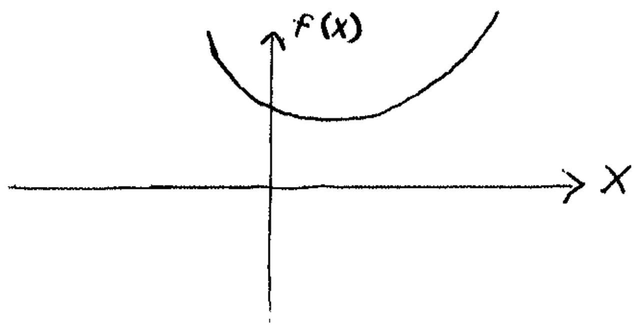
b) $\frac{x-5}{3} = \frac{y}{1} = \frac{z}{1}$

c) $\frac{x}{3} = \frac{y}{1} = \frac{z}{1}$

d) $\frac{x}{-2} = \frac{y}{5} = \frac{z}{1}$

e) $x = \frac{z+3}{2}, y = 0$

25. The curve drawn in the figure is the function $f(x) = Ax^2 + Bx + C$. Let $D = B^2 - 4AC$, then



a) $A > 0, \quad B > 0, \quad C > 0, \quad D < 0$

b) $A > 0, \quad B < 0, \quad C < 0, \quad D > 0$

c) $A < 0, \quad B > 0, \quad C > 0, \quad D > 0$

d) $A > 0, \quad B < 0, \quad C > 0, \quad D < 0$

e) $A < 0, \quad B > 0, \quad C < 0, \quad D > 0$

26. In order to determine the number of diagonals in a convex n-gon a student calculates the combination of n things (vertices) taken two at a time. She finds, however, that this calculation exceeds the actual number of diagonals in all cases by:

(a) 2 (b) n! (c) 2n (d) n (e) $\frac{n(n-3)}{2}$

27. The solution set for the equation $\tan(2\theta) + \cot(2\theta) = 2$ for angles θ such that $-\pi \le \theta \le 0$ is

a) $\{-\frac{3\pi}{8}, -\frac{7\pi}{8}\}$

b) $\{\frac{-3\pi}{4}, -\frac{7\pi}{4}\}$

c) $\{\frac{-9\pi}{8}, \frac{13\pi}{8}\}$

d) $\{\frac{\pi}{8}, \frac{9\pi}{8}\}$

e) $\{\frac{-\pi}{8}, \frac{-9\pi}{8}\}$

28. Which one of the following is a focus of the ellipse

$$\frac{x^2}{16} + \frac{(y+2)^2}{36} = 1 \quad ?$$

(a) $\left(2(1+\sqrt{5}), 0\right)$
(b) $\left(0, -2(1+\sqrt{5})\right)$
(c) $\left(2(1+\sqrt{5}), -2(1+\sqrt{5})\right)$
(d) $(2\sqrt{5}, -2\sqrt{5})$
(e) $(0, -2\sqrt{5})$

29. The equations of the line containing the point (-5, 0, 2) and perpendicular to the lines

$$\frac{x-2}{3} = \frac{y-3}{5} = \frac{z+1}{-4} \quad \text{and} \quad \frac{x+1}{1} = \frac{y-2}{4} = \frac{z+4}{-2} \text{ are}$$

a) $\frac{x+5}{6} = \frac{y}{2} = \frac{z-2}{7}$

b) $\frac{x+5}{-2} = \frac{y}{-1} = \frac{z-2}{2}$

c) $\frac{x+5}{6} = \frac{z-2}{7}$

d) $\frac{2x}{-1} = \frac{6y}{7} = \frac{z-2}{2}$

e) $\frac{x+5}{6} = \frac{y}{-2} = \frac{z-2}{7}$

30. Given $\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \ldots$ for $|x| < 1$, a power series representation of $\tan^{-1}x$ for $|x| < 1$ is

a) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$, $|x| < 1$

b) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$, $|x| < 1$

c) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{2n+1}$, $|x| < 1$

d) $\sum_{n=1}^{\infty}(-1) \frac{x^{2n-1}}{2n-1}$, $|x| < 1$

e) $\sum_{n=1}^{\infty}(-1)^{n+1} x^{2n-1}$, $|x| < 1$

31. The slope of the normal line to the curve $y = \ln[\cos^2 3x]$ at the point (0,0) is

a) 0

b) -6

c) $\frac{1}{6}$

d) $-\frac{1}{6}$

e) Undefined

32. How many terms are there in the expansion of $(a+b+c+d)^4$?

a) 4

b) 6

c) 28

d) 35

e) 64

33. Taking the well-known trigonometric identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, find

$$\sum_{k=1}^{n}[2\sin(kx)\sin(x/2)].$$

a) $\cos\left[\left(\frac{(2n-1)x}{2}\right)\right] - \cos\left[\left(\frac{(2n+1)x}{2}\right)\right]$

b) $\cos\left(\frac{nx}{2}\right) - \cos\left[\frac{(2n-1)x}{2}\right]$

c) $\cos\left(\frac{x}{2}\right) - \cos\left[\frac{(2n-1)x}{2}\right]$

d) $\cos\left(\frac{x}{2}\right) - \cos\left[\frac{(2n+1)x}{2}\right]$

e) 0

34. Let

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$ a, b where a and b are real numbers. $A^{10} =$

a) - I

b) A

c) -A

d) I

e) 0

35. Given two roots of the equation $2x^4 + x^3 + 5x^2 + 4x - 12 = 0$ are purely imaginary numbers. The other two roots must satisfy the equation

a) $2x^2 - x - 4 = 0$

b) $2x^2 - 2x + 1 = 0$

c) $2x^2 + x - 3 = 0$

d) $2x^2 + 3x - 2 = 0$

e) $2x^2 + 3x - 1 = 0$

36. The sequence $\{s_n\}$ given by $s_n = \frac{2^n}{n^2}$, $n = 1, 2, 3, \ldots$

a) Diverges

b) Has Limit 0

c) Has Limit 1

d) Has Limit 2

e) Has Limit $\frac{1}{2}$

37. $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 3x}\right) =$

a) $-\frac{3}{2}$

b) $\frac{3}{2}$

c) 0

d) ∞

e) $-\infty$

38. If A and B are square matrices of order n and X^t denotes the transpose of matrix X then $(AB)^t$ is

a) $A^t B^t$

b) $B^t A^t$

c) $\frac{1}{2}(A^t B + AB^t)$

d) $\frac{1}{2}(AB + BA)^t$

e) $(BA)^t$

39. The point on the graph of the function $f(x) = \sqrt{8x}$ which is nearest (3,0) is

a) (0,0)

b) $(1, 2\sqrt{2})$

c) $(\frac{1}{2}, 2)$

d) $(\frac{1}{8}, 1)$

e) $(\frac{7}{2}, 2\sqrt{7})$

40. The region bounded by the curve $y = ln(x)$, the x axis, and the line $x = e^2$ is revolved about the y axis. The volume of the solid generated is

a) $2\pi e^4$

b) $\frac{1}{2}\pi e^4$

c) $\frac{\pi}{2}(3e^4 + 1)$

d) $2\pi(e^4 - 3)$

e) 0