FORTY-THIRD ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics I 1999

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Scoring formula: 4R – W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

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1. Written without negative exponents, $(a^{-1}+b^{-1})^{-1} =$

a. $a+b$ b. $\frac{ab}{a+b}$ c. $\frac{a+b}{ab}$ d. $\frac{a^2+b^2}{a^2b^2}$ e. $\frac{a^2b^2}{a^2b^2}$

2. Which of the following statements is true?

I. The sum of two rational numbers is always a rational number.
II. The sum of a rational number and an irrational number is always an irrational number.
III. The sum of two irrational numbers is always an irrational number.

a. I only b. II only c. I and II only d. II and III only e. I, II, and III

3. The sum of the slopes of two lines which intersect at the point (1,1) is 5. If one of the lines passes through the point $(5, y_1)$ and the other line passes through $(5, y_2)$, then $y_1 + y_2 =$

a. 50 b. 22 c. 13 d. 6 e. $\sqrt{5}$

4. Let $f_0(x) = \frac{1}{x}$, $f_1(x) = (f_0 \circ f_0)(x)$. For positive integers $n > 1$, let $f_n(x) = f_0(f_{n-1}(x))$. Then $f_{10}(2) =$

a. $\frac{1}{2}$ b. $\frac{1}{1024}$ c. 2 d. 512 e. 1024

5. Suppose that a fair coin is tossed 4 times. Find the probability that the third head comes up on the fourth toss.

a. $\frac{3}{16}$ b. $\frac{3}{32}$ c. $\frac{1}{4}$ d. $\frac{1}{16}$ e. $\frac{1}{64}$

6. Solve the equation $\sin^{-1}(x) + \tan^{-1}\sqrt{3} = \frac{2\pi}{3}$ for x.

a. $x = 0$ b. $x = \frac{1}{2}$ c. $x = \frac{\pi}{6}$ d. $x = \frac{-\sqrt{3}}{2}$ e. $x = \frac{\sqrt{3}}{2}$

7. The range of the function $f(x) = e^{\cos x}$ is

a. $(-\infty, \infty)$ b. $[0, \infty)$ c. $[0, e]$ d. $\left[\frac{1}{e}, \infty\right)$ e. $\left[\frac{1}{e}, e\right]$

8. An otherwise smart person forgets the last digit of an important phone number. What is the probability the person dials the correct number on the fourth attempt?

a. $\frac{1}{10}$ b. $\frac{4}{10}$ c. $\left(\frac{1}{10}\right)^4$ d. $\frac{6}{10}$ e. $\frac{1}{(10)(9)(8)(7)}$

9. All forty people attending a political reception are either lawyers or politicians. Fifteen people at the reception are politicians. Ten people at the reception are both lawyers and politicians. How many people at the reception are lawyers?

a. 15 b. 20 c. 25 d. 30 e. 35

10. If both a and b are greater than 1 and $\log_b a = 3$, then $\log_b\left(\frac{a}{b}\right) =$

a. 1 b. 2 c. 3 d. -1 e. -3

11. Let $a_1 = 1$ and $a_n = a_{n-1} + 4$ for $n = 2,3,4,\cdots$. Evaluate the sum $a_1 + a_2 + a_3 + \cdots + a_{100}$.

a. 397 b. 401 c. 5050 d. 19,990 e. 94,109,400

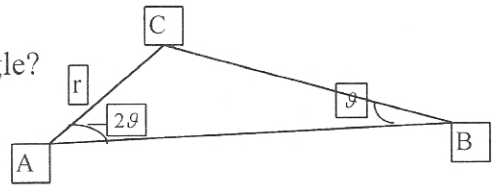
12. Let f be a function defined by $f(x) = \frac{1}{\tan\left(\frac{\pi}{2} - \frac{\pi}{x}\right)}$. Give all values of x in $[1, \infty)$ for which $f(x)$ is undefined.

a. $\{1\}$ b. $\{1,2\}$ c. $\{1,2,3,\cdots\}$ d. $\{2,4,6,\cdots\}$ e. $\{1,3,5,\cdots\}$

13. Suppose $a_n = \left(1 - \frac{1}{n+1}\right)\cos(n\pi)$ for $n = 0,1,2,3,\cdots$. Then $\lim_{n\to\infty} a_n =$

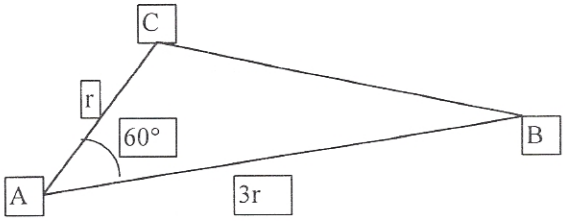
a. 1 b. $\{1,-1\}$ c. -1 d. ∞ e. does not exist

14. What is the length of side BC of the pictured triangle?



a. $2r$ b. $2r\sin\vartheta$ c. $\dfrac{r}{\tan\vartheta}$ d. $\dfrac{r}{\sin\vartheta}$ e. $2r\cos\vartheta$

15. What is the length of side BC of the pictured triangle?



a. r b. $r\sqrt{3}$ c. $r\sqrt{5}$ d. $r\sqrt{7}$ e. $2.5r$

16. Simplify the expression: $\sin^2(x)+\sin^2(y)-2\sin x\sin y\cos(x-y)$

a. $\cos(x+y)$ b. $\sin(x+y)$ c. $\sin^2(x-y)$ d. $\cos^2(x-y)$ e. $\sin(x-y)$

17. Find the value of $\sin^4 x+\cos^4 x$ if $\sin(2x)=\dfrac{2}{3}$.

a. $\dfrac{9}{7}$ b. $\dfrac{7}{9}$ c. $\dfrac{7}{8}$ d. $\dfrac{8}{7}$ e. $\dfrac{\sqrt{3}}{4}$

18. The exact value of $\sin\left(\tan^{-1}\left(-\dfrac{1}{2}\right)\right)$ is

a. $\dfrac{\sqrt{3}}{3}$ b. $\dfrac{-\sqrt{3}}{3}$ c. $\dfrac{\sqrt{5}}{5}$ d. $\dfrac{-\sqrt{5}}{5}$ e. undefined

19. If $f(x) = \arcsin(x+2)$ then

a. $f^{-1}(x) = \sin x - 2$

b. $f^{-1}(x) = \sin(x-2)$

c. $f^{-1}(x) = \sin x - 2, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

d. $f^{-1}(x) = \sin(x-2), \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

e. $f^{-1}(x) = \sin(x+2), \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

20. If the arithmetic sequence $\{a_n\}$ is such that $a_1 = 3$, $a_2 = 8$, $a_3 = 13$, $a_4 = 18$, $a_5 = 23$, etc., which term of the sequence has value 49938?

a. the 899th b. the 997th c. the 9987th d. the 9988th e. the 8998th

21. Written as an algebraic expression, $\cos(2\arccos(A))$ is

a. $1-2A^2$ b. $1+2A^2$ c. $2A^2-1$ d. $\sqrt{1-2A^2}$ e. $\sqrt{2A^2-1}$

22. How many y-intercepts does the graph of $y = \cos\left(\frac{1}{100x}\right)$ have?

a. zero b. one c. two d. three e. infinitely many

23. The quadratic function $f(x) = x^2 + bx + c$, where $b > 0$, and $0 < \sqrt{c} < \frac{b}{2}$ has

a. two distinct negative zeros
b. one positive zero and one negative zero
c. two distinct positive zeros
d. two repeated positive zeros
e. two repeated negative zeros.

24. The solution set of the equation $\log_2(2^x) + \log_2(4^x) + \log_{\frac{1}{2}}(2^x) = 4$ is

a. $\{0\}$ b. $\{0,1\}$ c. $\{2\}$ d. $\{0,2\}$ e. $\varnothing$ (the empty set)

25. A student failed to master any material for the upcoming quiz. The quiz will be a five question multiple-choice quiz with three choices for each question. If the student randomly picks an answer for each question, what is the probability that the student will get at least 3 correct answers?

a. $\frac{3}{5}$ b. $\frac{1}{27}$ c. $\frac{12}{27}$ d. $\frac{17}{81}$ e. $\frac{40}{243}$

26. If you travel from A to B at an average speed of 20 miles per hour and return from B to A along the same route at an average speed of 60 miles per hour, what is your average speed in miles per hour for the round trip?

a. 25 b. 30 c. $33\frac{1}{3}$ d. 35 d. 40

27. I went into a store and spent half of my money and then \$20 more. I went into a second store and spent half of my remaining money and then \$20 more. Than I had no money left. How much money did I have when I went into the first store?

a. $1\frac{1}{2}$ times the amount I had when I entered the second store

b. 2 times the amount I had when I entered the second store

c. 3 times the amount I had when I entered the second store

d. $3\frac{1}{2}$ times the amount I had when I entered the second store

e. 4 times the amount I had when I entered the second store

28. What is the maximum number of times that a horizontal line can intersect the graph of a fourth degree polynomial?

a. 1 b. 2 c. 3 d. 4 e. 5

29. Given the following data, select the true statement below.

x	f(x)	x	g(x)
0	3	0	3
1	6	1	6
3	12	2	12
7	24	3	24
15	48	4	48
31	96	5	96

a. f is an exponential function and g is a linear function.
b. f is a linear function and g is an exponential function.
c. f is a quadratic function and g is a linear function.
d. f is a linear function and g is a linear function.
e. f is an exponential function and g is an exponential function.

30. The graph of the function $f(x) = \dfrac{2x^2 + 2}{x}$ has

a. a vertical asymptote $x = 2$ and a horizontal asymptote $y = 2$

b. a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$

c. a vertical asymptote $x = 0$ and a horizontal asymptote $y = 2$

d. a vertical asymptote $x = 0$ and a slant asymptote $y = \dfrac{2}{x}$

e. a vertical asymptote $x = 0$ and a slant asymptote $y = 2x$

THE REMAINING PROBLEMS ARE PROBABLY BEST DONE WITH THE AID OF A GRAPHING CALCULATOR.

31. $100 is invested at an annual interest rate of 6% compounded monthly. How many months does it take for the investment to grow to $130?

a. 33 months b. 38 months c. 42 months d. 48 months e. 53 months

32. The domain of the function $f(x) = \log(\log(\log x)))$ is

a. $(-\infty, \infty)$ b. $(-\infty, 0)$ c. $(0, \infty)$ d. $(1, \infty)$ e. $(10, \infty)$

33. From a tank filled with 240 gallons of alcohol, 80 gallons are drawn off and the tank is filled up with water. Then 80 gallons of the mixture are removed and replaced with water, etc. How many gallons of alcohol remain after 23 drawings of 80 gallons each are made?

a. about .021 gallons
b. about .014 gallons
c. about $2.54 \cdot 10^{-9}$ gallons
d. about $1.70 \cdot 10^{-9}$ gallons
e. 0 gallons

34. How many solutions does the equation $5\cos(4x + 1) = 2$ have on the interval $0 \le x \le 2$?

a. 0 b. 1 c. 2 d. 3 e. 4

35. Find the sum of the first 20 terms of the geometric series $2 - 6 + 18 - 54 + \cdots$

a. -3,486,784,400
b. -1,162,261,466
c. -1,743,392,200
d. 1,743,392,200
e. 1,162,261,466

36. The town of Bedrock had 300 citizens in 1990. Assuming that the population increases by 4% each year, what will be population be in 2050, rounding to the nearest whole number?

a. 1020 b. 2918 c. 3035 d. 3156 e. 3282

37. Suppose a population P at time t is given by $P(t) = \dfrac{100{,}000}{1 + 99e^{-t}}$. Which of the following statements is true?

a. The population eventually reaches 1,000,000.
b. The population eventually reaches a maximum value and then begins to decrease.
c. The population increases over time, but never exceeds 100,000.
d. The population eventually goes to 0.
e. The population increases slowly at first, but after 9 years begins to increase faster.

38. How many real number solutions does the equation $x^5 - 100x^4 + x^3 - 300x^2 + x + 100 = 0$ have?

a. 1 b. 2 c. 3 d. 4 e. 5

39. The y-coordinate of the solution set of the system of equations $\begin{cases} y = \cos x \\ y = \ln x \end{cases}$ rounded to the nearest hundredth is

a. .24 b. .26 c. .28 d. .30 e. .32

40. What is the sum of the values of x such that $|x| + |x + 1| = 2$?

a. -1 b. 0 c. 1 d. 2 e. 3