FORTY-SECOND ANNUAL MATHEMATICS CONTEST
Sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics I
1998

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Scoring formula: 4R - W + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

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ADVANCED TOPICS I 1998

1. Let $f(x) = \frac{4x - 7}{2x + 3}$. If $f(g(x)) = x$, then $g(x)$ equals

a. $\frac{2x + 3}{4x - 7}$ b. $\frac{7 - 4x}{2x + 3}$

c. $\frac{2x + 7}{4x - 3}$ d. $\frac{3x + 7}{4 - 2x}$ e. $\frac{2x - 3}{4x + 7}$

2. If A and B are real numbers, then the graphs of $y = x^2 + Ax + B$ and $y = \log_5(x^2 + Ax + B)$ will both have no x-intercepts when

a. $A^2 - 4B > 0$ b. $A^2 - 4B < 0$

c. $A^2 - 4B = 0$ d. $A^2 - 4B + 4 > 0$ e. $A^2 - 4B + 4 < 0$

3. On a reviewing stand at a local army reserve parade, there are ten seats in the front row reserved for two colonels, three lieutenant colonels, and five captains. In how many different ways can the seats be labeled if they are labeled with ranks only?

a. 1,440 b. 2,520

c. 6,561 d. 80,640 e. 3,628,800

4. The graph of an exponential function of the form $y = Ae^{Bx}$ passes through the points (1, 2) and (5, 10). The value of A is

a. $\frac{\ln 5}{4}$ b. $\frac{2}{\sqrt[4]{5}}$

c. $\frac{2}{\sqrt[6]{5}}$ d. $\frac{8}{5}$ e. $\frac{2}{\sqrt[4]{8}}$

5. Let A be a 2 by 2 matrix with integer entries such that $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. How many distinct possibilities for A exist?

a. 1 b. 2

c. 4 d. 8 e. infinitely many

6. What is sin [2 (Arctan(A))]?

a. $\frac{2A}{\sqrt{1 + A^2}}$ b. $\frac{2A}{1 + A^2}$

c. $\frac{A}{\sqrt{1 + A^2}}$ d. $\frac{1}{\sqrt{1 + A^2}}$ e. $\frac{A}{1 + A^2}$

7. A circle of radius 5 units is centered at the origin in the xy-plane. A point P on the circle is located 60 degrees counterclockwise from the point (5, 0). Where does the line tangent to the circle at P intersect the x-axis?

a. (10, 0) b. $(5\sqrt{3}, 0)$

c. $(\frac{5 + \sqrt{3}}{2}, 0)$ d. (5, 0) e. $(10\sqrt{3}, 0)$

8. An investigator has a computer file showing incomes for 1,000 subjects. The incomes range from \$3,000 to \$90,000 a year. By accident the highest income in the file gets changed to \$900,000. The average income for the 1,000 subjects

a. is not affected b. increases by \$810

c. increases by \$900 d. increases by \$903 e. increases by \$10,000

9. A fair die in the shape of a cube has 4 green faces and 2 red faces. When the die is rolled repeatedly, which of the following sequences has the highest probability of occurring?

a. GGRGR b. GGRGRG

c. RRGRG d. RRGRGR e. RRRRG

10. A drawer contains a mixture of 6 identical black socks and 8 identical white socks. A person reaches into the drawer in the dark and randomly removes 2 socks. What is the probability that the socks are a matching pair?

a. $\frac{4}{13}$ b. $\frac{15}{91}$

c. $\frac{43}{91}$ d. $\frac{43}{93}$ e. $\frac{60}{1183}$

11. Let $f(x) = x^3 + Ax^2 + Bx - 5$, $f(1) = 2$ and $f(-1) = 4$. What is $3A - B$?

a. -14 b. 2

c. 6 d. 22 e. 26

12. Let A and B be real numbers with $|A| > 1$ and $|B| < 1$. Then $\sum_{k=2}^{\infty}\left(A^{3-k}B^{k+2}\right)$

a. equals $\frac{A^3B^3}{A - B}$ b. equals $\frac{A^2B^3}{1 - AB}$

c. equals $\frac{A}{A - B}$ d. equals $\frac{A^2B^4}{A - B}$ e. does not converge

13. Let P be the point (-6, 8) and Q be the point (6, 32). Which of the following points is 5/6 of the way from P to Q?

a. (4, 28) b. (-4, 12)

c. $(0, \frac{10}{3})$ d. (4, 12) e. (-4, 28)

14. Find the foci of the ellipse $9x^2 - 18x + 4y^2 + 16y = 11$.

a: $(0, \sqrt{5})$ and $(0, -\sqrt{5})$

b. (5, 0) and (-5, 0)

c. $(1 + \sqrt{5}, -2)$ and $(1 - \sqrt{5}, -2)$

d. d. $(1, -2 + \sqrt{5})$ and $(1, -2 - \sqrt{5})$

e. (2, 3) and (2, -3)

15. Find the fourth term of the infinite sequence defined recursively by $a_1 = 8$ and $a_{k+1} = 6k + a_k$.

a. 26 b. 44

c. 54 d. 62 e. 68

16. Which of the following is not a term in the expansion $(x^2 + 2y)^9$?

a. $5376x^6y^6$ b. $288x^8y^5$

c. $2016x^{10}y^4$ d. $672x^{12}y^3$ e. $144x^{14}y^2$

17. Two events A and B are independent. Let P(A) = 0.3 and P(B) = 0.4. Then $P(A \cup B)$ equals

a. 0.12 b. 0.88

c. 0.70 d. 0.58 e. 0.30

18. A special deck of cards consists of five red cards numbered 1 to 5, five blue cards numbered 1 to 5, and five white cards numbered 1 to 5. Three cards are selected at random without replacement from this special deck. Define A to be the event that the three cards selected are the same color. Define B to be the event that only even numbered cards are among the three cards selected. Define C to be the event that only red cards or cards numbered 5 are among the three cards selected. These three events in order from least likely to most likely to occur are

a. A, B, C b. B, A, C

c. C, B, A d. A, C, B e. B, C, A

19. The value of $\log_4 (\log_3(\log_2(512)))$ to the nearest hundredth is

a. -1.87 b. -0.36

c. 0.45 d. 0.50 e. 0.60

20. Let A be a real number with $A \neq 1$. Solving the equation $\log_A(\log_{A^2}(x)) = \log_{A^2}(\log_A(x))$ for x will yield

a. no solution b. 1 solution

c. 2 solutions d. 4 solutions e. infinitely many solutions

21. Let $H(x) = \frac{F(x)}{G(x)}$ where F(x) and G(x) are fourth degree polynomials with real coefficients which have no common factors. What is the maximum number of times the graph of y = H(x) can intersect its horizontal asymptote?

a. 0 b. 1

c. 2 d. 3 e. 4

22. Which of the following functions with specified domains given in radians will have an inverse function?

a. cos x with $\frac{\pi}{2} < x < \frac{3\pi}{2}$ b. sec x with $-\frac{\pi}{4} < x < \frac{\pi}{4}$

c. sin x with $\frac{3\pi}{4} < x < \frac{5\pi}{4}$ d. csc x with $\frac{\pi}{3} < x < \frac{5\pi}{6}$

e. cos x with $-\frac{\pi}{6} < x < \frac{\pi}{6}$

23. Chord AB divides the interior of a circle with diameter 4 centimeters into two regions. If chord AB is the perpendicular bisector of a radius of the circle, then the area in square centimeters of the smaller of these two regions is

a. $\frac{2\pi}{3} - \sqrt{3}$ b. $\frac{4\pi}{3} - \sqrt{3}$

c. $\frac{4\pi}{3} - 1$ d. $\frac{2\pi}{3} - 1$ e. $\frac{16\pi}{3} - 4\sqrt{3}$

24. For what integers A will the equation $x^3 - 5x^2 + Ax - 1 = 0$ have no rational solutions?

a. all integers except -7 and 7 b. all integers except -5 and 5

c. all integers except 5 and 7 d. all integers except -5 and 7

e. all integers except -7 and 5

25. If $\log_3 5 = A$, then the value of $\log_3\left(\frac{9}{25\sqrt{15}}\right)$ is

a. $\frac{5}{2} - \frac{1}{2}A$ b. $\frac{3}{2} - \frac{5}{2}A$

c. $\frac{5}{2} - \frac{3}{2}A$ d. $1 - 2A - \sqrt{A}$ e. $\frac{3}{2} - 2A - \sqrt{A}$

26. If $\tan 3t = \frac{\tan^3 t - 3(\tan t)}{3(\tan^2 t) - 1}$ and $\cos t = \frac{-1}{3}$ and $\sin t > 0$, then the value of

tan 3t is

a. $\frac{10\sqrt{2}}{23}$ b. $\frac{22\sqrt{2}}{23}$

c. $\frac{-22\sqrt{2}}{23}$ d. $\frac{-10\sqrt{2}}{23}$ e. $\frac{-23\sqrt{2}}{20}$

27. The median for the set of sample measurements {-10, -8, -2, 6, -4, 0, 11} is

a. -8 b. -2 c. -1 d. 0 e. 6

28. If you are hired at an annual salary of $18,000 and receive annual increases of 5%, what will your salary be when you begin your fifth year?

a. $20,795.18 b. $20,837.25

c. $21,879.11 d. $22,500.00 e. $22,973.07

29. If a pair of fair, standard six-sided dice is tossed, what is the probability the sum showing on the dice is a prime number?

a. $\frac{7}{18}$ b. $\frac{5}{12}$

c. $\frac{1}{5}$ d. $\frac{5}{36}$ e. $\frac{1}{12}$

30. How many x-intercepts does the graph of $y = \sin\left(\frac{1}{4x}\right)$ have?

a. zero b. one

c. two d. four e. infinitely many

31. A number of students are enrolled in a summer sports program. Each student has to participate in at least one of the three sports--swimming, tennis, and softball. If 5 students sign up for all 3 sports, 8 sign up for swimming and softball, 6 for swimming and tennis, 2 for softball and tennis but not swimming, 25 for softball, 20 for tennis and 21 for swimming, how many students are enrolled in the summer program?

a. 50 b. 55

c. 56 d. 66 e. 87

32. The graph of which of the following can include the points (1, -4), (4, -4), (1, 10), (4, 10) and (3, 5)?

a. a circle b. an ellipse

c. a hyperbola d. a parabola e. two intersecting lines

33. If the square matrix A is obtained from the square matrix B by adding three times the second row of B to the third row of B, then the determinant of A is equal to

a. the determinant of B
b. the negative of the determinant of B
c. three times the determinant of B
d. the negative of three times the determinant of B
e. three plus the determinant of B

34. If $x^2 + Ax + 5$ has no real zeros (roots), then A must satisfy

a. $A < \sqrt{20}$ b. $A = 4$

c. $A > \sqrt{20}$ d. $-\sqrt{20} < A < \sqrt{20}$ e. $A < -\sqrt{20}$ or $A > \sqrt{20}$

35. Let A and B be real numbers with $A^2 = B^2$. Then $(A + iB)^{10}$ equals

a. $A^{10} - B^{10}$ b. $A^{10} + B^{10}$

c. $(A^2 + B^2)^5 + i(A^2 + B^2)^5$ d. $(A^2 - B^2)^5 + i(A^2 - B^2)^5$

e. $i(32A^5B^5)$

36. Let A be a positive integer. If $\sum_{n=1}^{\infty}\left(\frac{420A}{n(n+A)}\right)$ is an integer, then A must satisfy

a. A is even b. A is odd

c. $A < 8$ d. $A > 420$ e. $8 \leq A \leq 420$

37. If $f(x) = 3x + 2$ and $g^{-1}(f^{-1}(x)) = 5x - 4$, then g(x) equals

a. $\frac{x - 6}{15}$ b. $\frac{5x - 6}{3}$

c. $\frac{x + 2}{15}$ d. $15x + 6$ e. $\frac{1}{15x + 6}$

38. The graph of $(x - y - 1)^2 + (x + y - 7)^2 = 0$ is

a. one point b. two points

c. two straight lines d. a parabola e. a circle

39. If the zeros of the quadratic $x^2 + Bx + C$ are both real and less than one and $S = B + C + 1$, then S must satisfy

a. $S < 0$ b. $S = 0$

c. $S > 0$ d. $-1 < S < 1$ e. $S > 1$

40. At 2:15 the hour hand and the minute hand of an analog clock form an angle of

a. 5° b. 7.5° c. 22.5° d. 28° e. 30°