

FORTIETH ANNUAL MATHEMATICS CONTEST

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THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics I 1996

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Scoring formula: $4R - W + 40$

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft head (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you. If you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly 80 minutes to work.

Contributors to TMTA for the Annual Mathematics Contest:

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ADVANCED TOPICS I

1. Let s denote the sum of the infinite series

$$\sum_{k=0}^{\infty} \frac{\cos k\pi}{2^k}$$

Then s is:

- a) 0;
- b) $2/3$;
- c) 1;
- d) 2;
- e) not defined.

2. If $0 < \alpha < \pi$ and $\cos \alpha = \frac{\sqrt{2}}{2}$, then $\tan \alpha$ is:

- a) 0;
- b) $\frac{\sqrt{2}}{2}$;
- c) 1;
- d) $\frac{2}{\sqrt{2}}$;
- e) 2.

3. If $y = \sqrt{\sin^2 x + \sin^2(\pi/2 - x)}$, then y is:

- a) 0;
- b) $\sqrt{2} \sin x$;
- c) $\sqrt{2} \cos x$;
- d) $\sin x + \sin(\pi/2 - x)$;
- e) 1.

4. The period of the function $f(x) = |\sin(3x + 2)|$ is:

- a) 2π ;
- b) $\frac{2\pi}{3}$;
- c) π ;
- d) $\frac{\pi}{3}$;
- e) 6π .

5. The graph of $f(x) = 2e^{x+3}$ can be determined by one or more of the following operations:

- (i) shifting the graph of $g(x) = 2e^x$ three units to the left.
- (ii) shifting the graph of $g(x) = 2e^x$ three units to the right.
- (iii) vertically expanding the graph of $g(x) = 2e^x$ by a factor of 3 units.
- (iv) vertically expanding the graph of $g(x) = 2e^x$ by a factor of e units.

Which of these is possibly correct?

- a) (i) only;
- b) (ii) only;
- c) (iii) only;
- d) (ii) and (iii) only;
- e) (i) and (iv) only.

6. Suppose that f is a function with the following properties:

- (a) the range of f is $[-12, 17]$
- (b) f is an even function

(c) f assumes the given values:

x	-9	-4	-2	0	3	10
$f(x)$	5	14	9	-1	4	-3

Indicate which of the following statements must be true.

- a) $f(-3) = 5$;
- b) 14 is an maximum value of f ;
- c) If $h(x) = f(2x)$ then $h(5)$ is negative;
- d) $f(-x) = -f(x)$ for all x in the domain of f ;
- e) the graph of f crosses the x -axis twice.

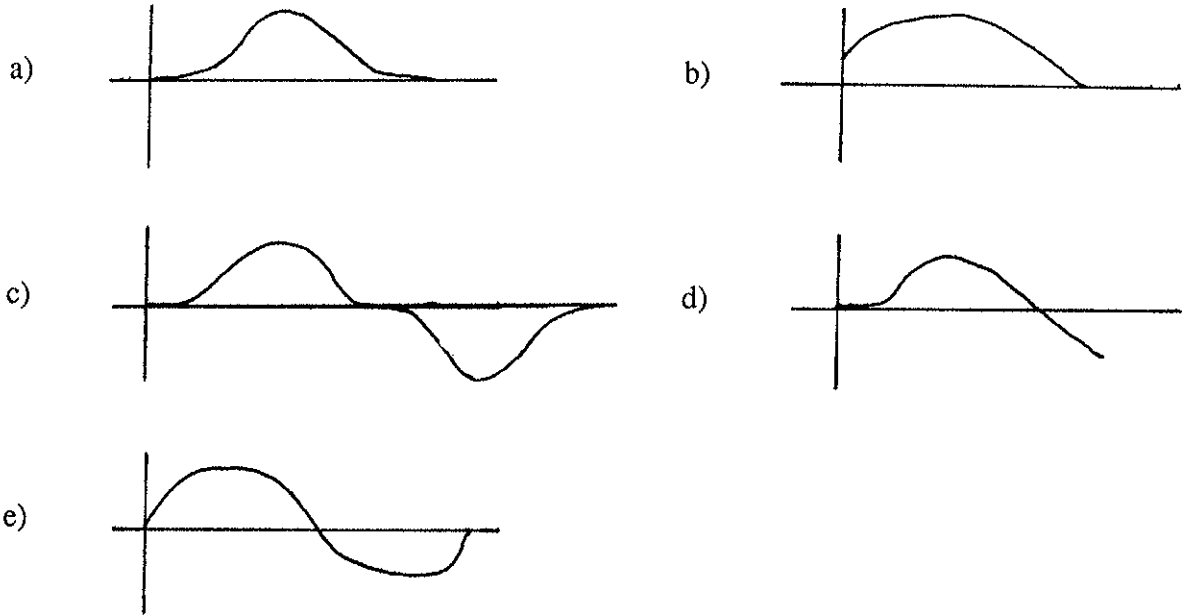
7. The equation $x^5 - 3x^3 + 4x^2 + x - 2 = 0$ has the following number of root(s) or zero(s):

- a) no rational roots;
- b) one rational root;
- c) two rational roots;
- d) three rational roots;
- e) four rational roots.

8. The equation $-3^{x+1}(2x+5)(x-7)^3 \ln(x-4)(3x-1)^{-\frac{1}{2}} = 0$ has as its solution set:

- a) $\{-1, -\frac{5}{2}, 7, 4, \frac{1}{3}\}$;
- b) $\{-\frac{5}{2}, 7, 4, \frac{1}{3}\}$;
- c) $\{-\frac{5}{2}, 7, 4\}$;
- d) $\{-\frac{5}{2}, 7, 5, \frac{1}{3}\}$;
- e) $\{-\frac{5}{2}, 7, 5\}$.

9. An airplane takes off from one airport and 45 minutes later, lands at another airport 300 miles away. Let t represent the time in minutes since the plane took off and let $y(t)$ represent the altitude of the plane t minutes after takeoff. Which of the graphs below best represents the graph of the plane's vertical velocity as a function of time?

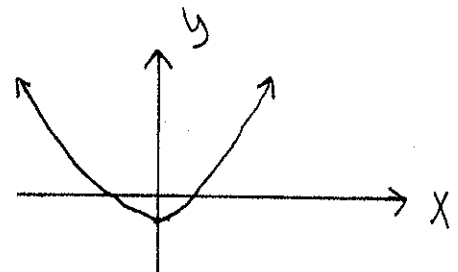


10. Find the standard form of the equation of the circle passing through $(x, y) = (-1, 2)$ if its area is 9π .

- a) $(x + 2)^2 + (y - 2)^2 = 9$;
- b) $(x + 1)^2 + (y - 2)^2 = 9\pi$;
- c) $(x - 2)^2 + (y + 1)^2 = 3$;
- d) $x^2 + y^2 = 5$;
- e) not enough information.

11. If the graph of the function $y = cx^2 + b$ has the form then:

- a) $c > 0, b > 0$;
- b) $c < 0, b > 0$;
- c) $c > 0, b < 0$;
- d) $c = 0, b < 0$;
- e) $c > 0, b = 0$.



12. Find the equation of the ellipse with center at the origin whose major axis is vertical if the ellipse is passing through (0, 4) and (3, -2).

a) $\frac{x^2}{12} + \frac{y^2}{16} = 1;$

b) $\frac{x^2}{9} + \frac{y^2}{4} = 1;$

c) $\frac{x^2}{16} + \frac{y^2}{13} = 1;$

d) $\frac{x^2}{9} + \frac{y^2}{36} = 1;$

e) $y = -2x + 4.$

13. Given $\cos 2x = \frac{3}{5}$, $\sin^4 x - \cos^4 x$ is:

a) $\frac{3}{5};$

b) $\frac{2}{5};$

c) $-\frac{3}{5};$

d) $\frac{4}{25};$

e) $\frac{9}{25}.$

14. If $\sin(60^\circ - x)(\cos x + \sqrt{3} \sin x) = 1$ and x is an acute angle, then $x =$

a) $60^\circ;$

b) $45^\circ;$

c) $30^\circ;$

d) $15^\circ;$

e) $0^\circ.$

15. The function $y = 3 \sin 3x \cos 3x$ attains its maximum value at:

a) $\frac{\pi}{12};$

b) $\frac{\pi}{10};$

c) $\frac{\pi}{8};$

d) $\frac{\pi}{6};$

e) $\frac{\pi}{4}.$

16. The median salary of 10 employees at the "New-and-Better" company was \$50,000, and their mean salary was \$45,000. Then, only the highest paid employee got a raise of \$5,000. What is the sum of the new (after the raise) median and mean salaries?
- a) \$95,500;
 - b) \$96,000;
 - c) \$100,000;
 - d) \$105,000;
 - e) not enough information to answer the question
17. Of the 30 mathematics majors at the Wiseman College, 10 are men and 20 are women. Of the 10 men math majors, 4 are seniors. Of the 20 women math majors, 5 are seniors. If a math major is chosen at random, what is the probability that the student is a woman or a senior?
- a) $1/6$;
 - b) $1/4$;
 - c) $1/2$;
 - d) $4/5$;
 - e) $29/30$.
18. There are 4 red and 8 black balls in a box. Two balls are drawn from the box without replacement. What is the probability that at least one of them is red?
- a) $14/33$;
 - b) $16/33$;
 - c) $17/33$;
 - d) $19/33$;
 - e) $20/33$.
19. A pair of fair dice is tossed. The faces on each die are numbered 1 through 6. What is the probability that the total on the two dice is at most 4?
- a) $1/12$;
 - b) $1/9$;
 - c) $5/36$;
 - d) $1/6$;
 - e) $7/36$.
20. I planted 10 trees in my garden this summer. The variance of their height is $3m^2$. After 10 years my trees will (hopefully) double their height. What will be the variance of the heights of my 10 trees after 10 years?
- a) $3m^2$;
 - b) $6m^2$;
 - c) $12m^2$;
 - d) $60m^2$;
 - e) not enough information to answer the question

21. In a small town 40 percent of voters (those eligible to vote) are Liberals, and 60 percent are Conservatives. Records show that in the last election 55 percent of the Liberals voted, and 35 percent of the Conservatives voted. If a person eligible to vote is selected at random from that town, what is the probability that he/she voted in the last election?

- a) 0.4;
- b) 0.43;
- c) 0.47;
- d) 0.65;
- e) 0.9.

22. A restaurant offers six sandwiches, three cold drinks, and five desserts as part of its luncheon special. How many different specials (one sandwich, one drink, one dessert) are there?

- a) $6 + 3 + 5$;
- b) $6! + 3! + 5!$;
- c) $6!3!5!$;
- d) 90;
- e) 15.

23. Consider a set of 10 people. Suppose one and only one is named Harry. How many committees of size 4 can be formed from the given 10 people if you insist Harry be a member of the committee?

- a) $\binom{9}{3}$;
- b) $\binom{10}{4}$;
- c) $\binom{10}{3}$;
- d) $\binom{9}{4}$;
- e) $9!$.

24. Find the percent of increase (to the nearest whole number) in the volume of a sphere when the surface area of the sphere is increased by 25%.

- a) 50%;
- b) 40%;
- c) 60%;
- d) 75%;
- e) 30%.

25. Find $(e^{2\ln x})^2$.

- a) $2x^2$;
- b) $4x^2$;
- c) e^{2x^2} ;
- d) e^{x^4} ;
- e) x^4 .

26. The solution of the equation $\log_{10} 2 + \log_{10} x = \ln e$ is:

- a) 5;
- b) e;
- c) 10;
- d) e/2;
- e) $\log_{10} e$.

27. How long will it take to triple \$200 if it is invested at the interest rate 8% compounded quarterly?

- a) 25 years;
- b) $\frac{\ln 3}{4 \ln 1.02}$ years;
- c) e^3 years;
- d) $\frac{4 \ln 3}{\ln 1.02}$ years;
- e) $\ln 3$ years.

28. Solve the equation $3 = \frac{10^x - 10^{-x}}{2}$ for x.

- a) $\log_{10} 6$;
- b) $\ln(3 - \sqrt{10})$;
- c) $\log_{10}(3 + \sqrt{10})$;
- d) $\log_{10}(3 - \sqrt{10})$;
- e) $3 + \sqrt{10}$ and $3 - \sqrt{10}$.

29. Which of the following equations is a polar coordinate representation of the parabola $y = x^2$.

- a) $r = \sec \theta$
- b) $r = \tan \theta$
- c) $r \cos \theta = 1$
- d) $r = \sec \theta \tan \theta$
- e) $r^2 = \sin \theta$

30. The solution set of the inequality $-2|x + 1| \geq 2$ is:

- a) $(-\infty, -2]$;
- b) $[-2, 0]$;
- c) $[-2, \infty)$;
- d) empty;
- e) all real numbers.

31. The solution set of $\frac{1}{x-1} \leq \frac{1}{x+1}$ is:

- a) $(-1, 1)$;
- b) $(-\infty, -1) \cup (1, \infty)$;
- c) empty set
- d) $(-\infty, \infty)$;
- e) $[-\frac{1}{2}, \frac{1}{2}]$.

32. Consider $f(x) = ax^2 + bx + c$, where a , b , and c are odd integers. Let r and s be the roots of f . Which one of the following statements about r and s is false?

- a) Neither r nor s is an integer;
- b) $r + s = -\frac{b}{a}$;
- c) $r \cdot s = \frac{c}{a}$;
- d) both r and s can be rational numbers;
- e) $r \neq s$.

33. If a complex number z satisfies the equation $z^2 - 2z + 2 = 0$ then z must be:

- a) $1 + i$;
- b) $\pm 1 + i$;
- c) $1 - i$;
- d) $1 \pm i$;
- e) $\pm 2i$.

34. If $(1 + z)^2 = -4$, then $|z|$ is:

- a) -5 ;
- b) 1 ;
- c) $-1 + 2i$;
- d) $-1 - 2i$;
- e) $\sqrt{5}$.

35. Let $f(t) = t^3$ and $g(t) = 3t + 1$. Find the inverse of the composition, that is, find $(f \circ g)^{-1}(t)$.

- (a) $\frac{\sqrt[3]{t-1}}{3}$ (b) $\frac{\sqrt[3]{t-1}}{3}$ (c) $\sqrt[3]{\frac{t-1}{3}}$ (d) $(3t + 1)^{-3}$ (e) $\frac{1}{3t^3 + 1}$

36. The graph of $2x^2 + 2y^2 + 4x - 8y = 8$ is :

- a) a parabola;
- b) an ellipse;
- c) a straight line;
- d) a circle with center at $(-1, 2)$ and radius 3;
- e) a circle with center at $(-1, -2)$ and radius 2.

37. Find the constant c such that $r(x) = (x + 2)^2 - (x - c)^2$ is symmetrical with respect to the origin.

- (a) $c = 0$;
- (b) $c = x$;
- (c) $c = 2$ or $c = -2$;
- (d) $c = -1$;
- (e) $c = 1$ or $c = -1$.

38. Given $f(3x - 1) = 9x^2 - 1$ for all x , then $f(x)$ is

- (a) $9(x + 1)^2 - 1$;
- (b) $9(3x - 1)^2 - 1$;
- (c) $9(3x + 1)^2 - 1$;
- (d) $\frac{(x+1)^2}{9} - 1$;
- (e) $(x + 1)^2 - 1$.

39. Assume $f(x) = 2x - 3$ and $g(x) = ax + b$, where a and b are some real numbers. If $f(g(x)) = g(f(x))$, then

- (a) $a = 0, b = 0$;
- (b) $a = 2, b = 3$;
- (c) $2a - 3 = 0$;
- (d) $3a + b = 3$;
- (e) $a + 3b = 1$.

40. If $f(x) = \frac{ax - 5}{4x + 7}$ and $f(f(f(x))) = f(x)$, then

- (a) $a = -5$;
- (b) $a = 4$;
- (c) $a = 7$;
- (d) $a = -7$;
- (e) a can be any real number.