THIRTY-FIFTH ANNUAL MATHEMATICS CONTEST
sponsored by
THE TENNESSEE MATHEMATICS TEACHERS' ASSOCIATION

Advanced Topics I 1991

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Scoring formula: 4R - W + 40

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DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer, and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your very best on the questions you feel you know how to work. You will be penalized for incorrect answers, so it is advisable not to do wild guessing.

If you should change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on your answer sheet. The answer sheets will not be returned to you. If you wish to have a record of your performance, mark your answers in this booklet also. You will be able to keep this booklet after the test is completed.

When told to do so, open your test booklet and begin. The working time for the entire test is 80 minutes.

Contributors to TMTA for Annual Mathematics Contest:

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ADVANCED TOPICS I

1. Find the distance between the x-intercept and the y-intercept of the line $y = -3x + 6$.

   a) $2\sqrt{10}$ b) $5\sqrt{2}$ c) $3\sqrt{3}$ d) 6 e) 7

2. For what value of k does the line $kx + 5y = 7$ have a slope of 3?

   a) -15 b) -11 c) $-\frac{3}{5}$ d) $-\frac{5}{3}$ e) $\frac{5}{3}$

3. If $(x + y)^2 = 4$ and $(x - y)^2 = 12$ then $xy =$

   a) -8 b) -5 c) -4 d) -2 e) -3

4. Determine the area of the right triangle pictured below.

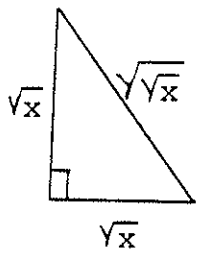
   a) 1/8

   b) 1/6

   c) 1/4

   d) 1/2

   e) 1



5. In a right triangle, the shortest leg is 4 units shorter than the other leg and 8 units shorter than the hypotenuse. Find the area of the triangle.

   a) 60 b) 80 c) 90 d) 92 e) 96

6. $\sqrt[3]{\sqrt[4]{9}} \cdot \sqrt[4]{\sqrt[3]{16}} =$

   a) $\sqrt[7]{144}$ b) $\sqrt[7]{12}$ c) $\sqrt[6]{12}$ d) $\sqrt[12]{25}$ e) $\sqrt[12]{6}$

7. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right)^{30} =$

   a) -1 b) $\sqrt{2} + \sqrt{2}\,i$ c) $2\sqrt{2} - \sqrt{2}\,i$ d) $-i$ e) 1

8. The graphs of $4x^2 + 4y^2 = 36$ and $4x^2 + 9y^2 = 36$ intersect in exactly _____ point(s).

a) 0 b) 1 c) 2 d) 3 e) 4

9. Given that $\ln(2 + \sqrt{3}) \doteq 1.31696$, which of the following values of x and y satisfy $\ln(x + y) \doteq -1.31696$?

a) $x = -2,\ y = -\sqrt{3}$

b) $x = -2,\ y = \sqrt{3}$

c) $x = 3,\ y = -\sqrt{2}$

d) $x = 2,\ y = -\sqrt{3}$

e) $x = -3,\ y = -\sqrt{2}$

10. A card is selected at random from a standard deck of 52 playing cards and then a second card is selected at random from the deck without replacing the first card. Find the probability that the first card is a king and the second card is a heart.

a) $\frac{1}{52}$ b) $\frac{3}{104}$ c) $\frac{1}{51}$ d) $\frac{4}{221}$ e) $\frac{7}{104}$

11. A man has one one-dollar bill, one two-dollar bill, one five-dollar bill, one ten-dollar bill, one twenty-dollar bill, one fifty-dollar bill, and one hundred-dollar bill. If he gives his son at least two of these bills, how many different choices are possible?

a) 98 b) 100 c) 120 d) 110 e) 248

12. A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line through the point (1, 2) (see figure). Write the area as a function of x and determine the domain of the function.

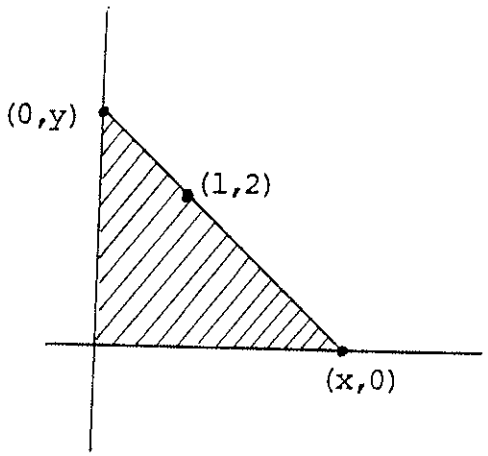
a) $A = \frac{x^2}{x^2 - 1}, \quad x > 0$

b) $A = \frac{x^2}{x + 1}, \quad x > 0$

c) $A = \frac{2x}{x - 1}, \quad x > 1$

d) $A = \frac{x^2}{x - 1}, \quad x > 1$

e) $A = \frac{2x^2}{x^2 + 1}, \quad x > 1$



13. Which of the following sets of numbers can be the lengths of sides of a right triangle? (Here, $.\overline{4}$ means .444...).

a) $.\overline{1},\ .\overline{4},\ .\overline{9}$

b) $\sqrt{.\overline{1}},\ \sqrt{.\overline{4}},\ \sqrt{.\overline{9}}$

c) $\sqrt[4]{.\overline{1}},\ \sqrt[4]{.\overline{4}},\ \sqrt[4]{.\overline{9}}$

d) $(.\overline{1})^2,\ (.\overline{4})^2,\ (.\overline{9})^2$

e) $(.\overline{1})^4,\ (.\overline{4})^4,\ (.\overline{9})^4$

14. Find the distance from the point (3, 2) in the Cartesian plane to the point on the graph of $y = \sqrt{x} + 2$ that lies closest to (3, 2).

a) $\frac{1}{2}\sqrt{11}$ b) $\frac{3}{2}$ c) 1 d) 2 e) $\sqrt{3}$

15. If $f(x) = x^3 - 5$ and $(f \circ g)(x) = \sqrt{x - 2}$ then $g(11) =$

a) 0 b) 1 c) 2 d) 3 e) 4

16. In the figure below, $\overline{AB} \perp \overline{BC}$, $\overline{BD} \perp \overline{AC}$, AD = p and DC = q. Then the area of $\triangle ABC$ is

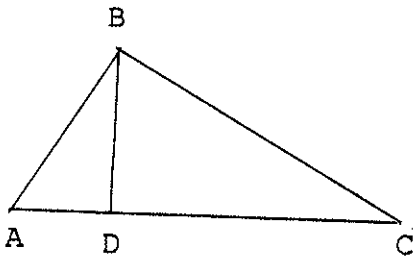
a) 2 pq

b) $(p + q)\sqrt{pq}$

c) $\frac{p + q}{2}\sqrt{pq}$

d) $(p + q)\sqrt{p + q}$

e) $p^2 + q^2$



17. An antenna mounted on top of a vertical pole is observed from a point 30 feet from the base of the pole and the angle of elevation to the bottom of the antenna is 50° while the angle of elevation to the top of the antenna is 53°. How tall is the antenna?

a) 30(tan 50° - tan 53°) feet

b) 30(cot 50° - cot 53°) feet

c) 30(tan 53° - tan 50°) feet

d) 30(cot 53° - cot 50°) feet

e) 30(tan 40° - tan 37°) feet

18. If a + b = 11, b + c = 14 and c + d = 19, then a + d =

a) 15 b) 16 c) 17 d 18 e) 19

19. If $\sin x + \cos x = 1/2$ then $\sin^3 x + \cos^3 x =$

a) $\frac{5}{16}$ b) $\frac{11}{16}$ c) $\frac{1}{2}$ d) $\frac{1}{8}$ e) $\frac{1}{4}$

20. Which function does not have the property that $f(f(x)) = x$?

a) $f(x) = -x$

b) $f(x) = 1/x$

c) $f(x) = \sqrt[3]{4 - x^3}$

d) $f(x) = \ln(2 - e^x)$

e) $f(x) = \sqrt{x}$

21. If $\frac{1}{x} + \frac{1}{y} = 5$ and $x + y = 6845$ then $\sqrt{xy} =$

a) 29 b) 31 c) 35 d) 37 e) 39

22. If $x = \log_{2\sqrt{2}} 64$, find x.

a) 4 b) 18 c) 8 d) 3 e) 6

23. Hexagon ABCDEF is regular, G is the midpoint of $\overline{CF}$ and $GH = 5\sqrt{3}$. Find GC.

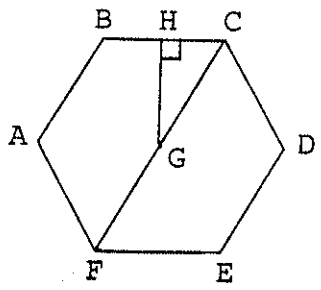
a) 15

b) 10

c) $6\sqrt{2}$

d) $5\sqrt{6}$

e) $10\sqrt{3}$



24. Let r_1 and r_2 be the roots of $1776x^2 + 1990x - 1492 = 0$. Then $\frac{1}{r_1} + \frac{1}{r_2} =$

a) $\frac{1}{2}$ b) $\frac{1990}{1776}$ c) $\frac{1776}{1990}$ d) $\frac{1990}{1492}$ e) 1

25. If $125^{a+2b} = 25$ and $\left(\frac{1}{3}\right)^{-2a-8b} = 9$ then ab =

a) $\frac{1}{18}$ b) $\frac{1}{6}$ c) 4 d) $-\frac{1}{18}$ e) $-\frac{1}{6}$

26. The two circles pictured below are concentric, $\overline{AB}$ is tangent to the smaller circle, and AB = 62. What is the area of the region which is outside the smaller and inside the larger circle?

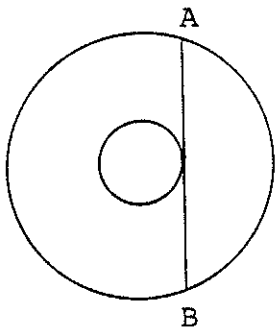
a) 62π square units

b) 3844 square units

c) 3894π square units

d) 620π square units

e) 961π square units



27. If $f(x) = \sqrt{x^2 + 50x + 625} - \sqrt{x^2 - 66x + 1089}$ then $f(-381.67) =$

a) 25 b) 126.33 c) 58 d) -58 e) -126.67

28. The sum of the first two terms of a geometric sequence is 9 and the sum of the first six terms of that sequence is 279. What is the sum of the first four terms of the sequence?

a) 45 b) 54 c) 55 d) 57 e) 59

29. If $f(x) = \sqrt{3x - 1}$ for $x \geq 1/3$, then $f^{-1}(x)$, the inverse of f, would be

a) $\frac{1}{\sqrt{3x - 1}}$ b) $9x^2 - 6x + 1$ c) $\frac{x^2 + 1}{3}$ d) $\frac{x^3 + 1}{2}$ e) $\frac{x^2}{3} + 1$

30. $\sqrt{20 + \sqrt{20 + \sqrt{20 + \ldots}}} =$

a) 5 b) -4 c) 4.7 d) 5.1 e) 4

31. A geoboard contains 25 pegs arranged in a rectangular grid with 5 equally spaced rows and columns. How many different squares with all 4 corners at pegs can be formed by stretching a rubber band around the pegs? Three such squares are shown below.

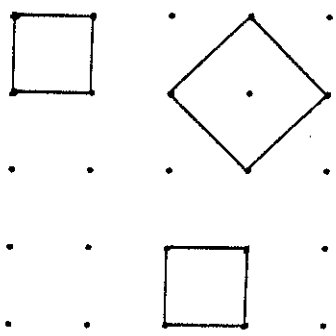
a) 30

b) 34

c) 39

d) 46

e) at least 50



32. How many equilateral triangles with all vertices at pegs can be formed on a geoboard like the one in problem 31?

a) 0 b) 4 c) 12 d) 28 e) 54

33. If matrix $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ then the inverse A^{-1} is

a) not defined b) $\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ c) $\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}$ d) $\begin{bmatrix} -3 & 2 \\ -4 & 3 \end{bmatrix}$ e) $\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$

34. Which of the following is a true statement?

a) $2^{1/6} < 3^{1/10} < 6^{1/15}$

b) $2^{1/6} < 6^{1/15} < 3^{1/10}$

c) $3^{1/10} < 6^{1/15} < 2^{1/6}$

d) $3^{1/10} < 2^{1/6} < 6^{1/15}$

e) $6^{1/15} < 3^{1/10} < 2^{1/6}$

35. If $\sin^2 \frac{x}{9} + \sin^2 \frac{2x}{9} + \sin^2 \frac{3x}{9} + \sin^2 \frac{4x}{9} = \frac{9}{4}$ then $\cos^2 \frac{x}{9} + \cos^2 \frac{2x}{9} + \cos^2 \frac{3x}{9} + \cos^2 \frac{4x}{9} =$

a) $\frac{9}{4}$ b) 2 c) $\frac{13}{8}$ d) $\frac{7}{4}$ e) $\frac{4}{3}$

36. Find the sum of those x-values such that $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan^2 x + (1 - \sqrt{3}) \tan x - \sqrt{3} = 0$.

a) $\frac{\pi}{3}$ b) $\frac{13}{12}\pi$ c) $\frac{5}{3}\pi$ d) $\frac{\pi}{12}$ e) $\frac{25}{6}\pi$

37. Tom and Sue are playing a game where they take turns tossing a coin and the first one to get "heads" wins. If Tom tosses first, what is the probability that he will win the game?

a) $\frac{1}{2}$ b) $\frac{1}{4}$ c) $\frac{3}{4}$ d) $\frac{7}{8}$ e) $\frac{2}{3}$

38. Beginning with ΔABC, the squares ADEB, BFGC and CHIA are drawn in, having one side in common with ΔABC. Then corners of the squares are connected to form the shaded triangles having areas A_1, A_2 and A_3. If AC > BC > AB then which of the following is true?

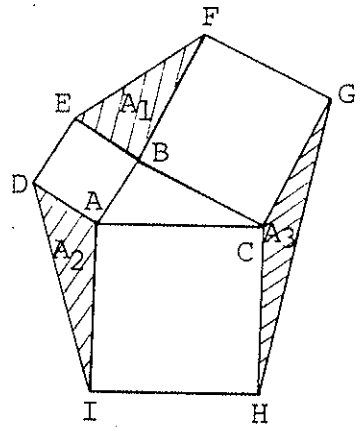
a) $A_1 > A_2 > A_3$

b) $A_2 > A_1 > A_3$

c) $A_3 > A_1 > A_2$

d) $A_3 > A_2 > A_1$

e) $A_1 = A_2 = A_3$



39. A white square is divided into 4 smaller congruent squares and the square in the upper left hand corner is shaded in. Then each of the remaining 3 white squares is divided into 4 smaller congruent squares and the one in the upper left hand corner is shaded in. This process is continued where each step involves dividing each of the remaining white squares into 4 smaller squares and shading in the smaller square in the upper left hand corner. How many different sizes of squares must be shaded in before at least two-thirds of the original square has been shaded in?

a) 3 b) 4 c) 5 d) 6 e) 67

40. Given matrix A of m x n dimension and matrix B of p x q dimension and the following statements:

I. For A + B to exist, m = p and n = q.
II. For A • B to exist, n = p.
III. For B • A to exist, A • B must exist.
IV. For A • B to exist, m = n and p = q.

Which of these statements is always true?

a) III only b) IV only c) II and IV d) I, II, and IV e) I and II