TENNESSEE MATHEMATICS TEACHERS ASSOCIATION

SIXTIETH ANNUAL MATHEMATICS CONTEST

2016

Algebra II

Prepared by:

Department of Mathematics & Statistics
The University of Tennesse at Martin
Martin, Tennessee

Reviewed by:

Mathematics Faculty
Carson-Newman University
Jefferson City, Tennessee

Coordinated by: Louis Kolitsch

Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

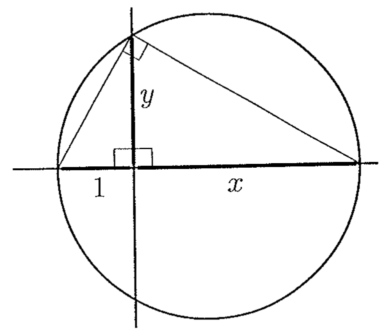
When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

1. What is the largest number of cubes with edges of length 2 cm that can be placed in a cube with edges of length 20 cm?

   A. 1 B. 10 C. 1000 D. 10000 E. 100000

2. In the geometric figure on the right, the diameter of the circle is $1+x$. Determine the length y.

   A. 2 B. $\frac{x}{1+x}$ C. $\frac{1+x}{x}$ D. $\sqrt{x}$ E. $\sqrt{1+x}$



3. John has a set of 2016 numbers, and the mean of those numbers is 2016. John removed one number and the remaining numbers had a mean of 2015. What was the value of the number that John removed?

   A. 2016−2015 B. 2016/2015 C. 2016+2015 D. 2016·2015 E. 2015^{2016}

4. Which of the following is NOT equal to one for all real numbers x?

   A. $\sin^2 x + \cos^2 x$ B. $\sec^2 x - \tan^2 x$ C. $\frac{x^2+1}{x^2+1}$ D. $(x+1)^2 - x(x+2)$ E. $(x-x)!$

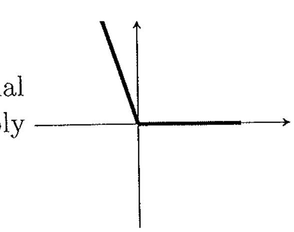
5. Determine the product $(2+i)(3-4i)(5+2i)$.

   A. $30-8i$ B. $60-5i$ C. $20-i$ D. $48-33i$ E. 38

6. The sum of the four zeros of $2x^4 + 7x^3 + 4x^2 + 2x - 3$ is

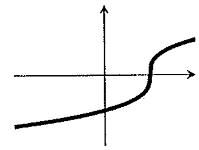
   A. −7/2 B. 12 C. 7/2 D. 5/2 E. 43

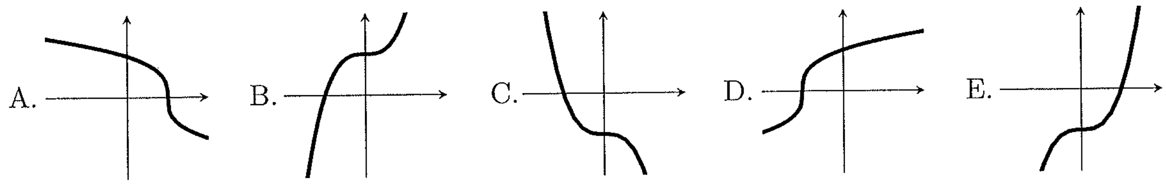
7. The image to the right shows an angle in standard position with the terminal side in the second quadrant. Which of the following might reasonably approximate the radian measure of this angle?



   A. 1 B. 1.5 C. 2 D. 2.5 E. 3

8. The graph of $y = f(x)$ is shown to the right. Which of the graphs below is the graph of $y = f^{-1}(x)$?



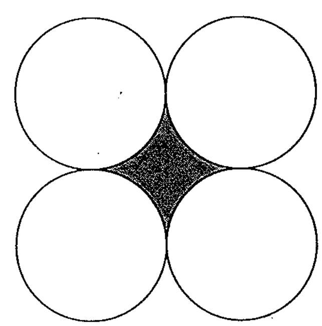


9. The polynomial $x^4 + 64$ factors as
   A. $(x^2 + 8)^2$
   B. $(x^2 + 8)(x^2 - 8)$
   C. $(x^2 + 2x + 4)(x^2 - 8x + 16)$
   D. $(x^2 - 4x + 8)(x^2 - 4x - 8)$
   E. $(x^2 - 4x + 8)(x^2 + 4x + 8)$

10. Together one brick and one-third of a brick weigh 13 pounds more than one-fourth of a brick. How many pounds does one brick weigh?
    A. 10.25   B. 12   C. 13   D. 12.325   E. 11.676

11. Suppose a sequence is defined by $a_0 = 2$ and $a_{n+1} = a_n^2 - 5$ for $n \geq 0$. What is the value of the first term in this sequence that is greater than 10000, if such a term exists?
    A. 11600
    B. 13451
    C. 235266
    D. 913931
    E. No term in this sequence is greater than 10000.

12. The tangent circles in the drawing on the right all have a radius of one unit. What is the area of the shaded region?
    A. $4-\pi$ B. 2π C. $\pi/4$ D. $\sqrt{3}-\frac{\pi}{2}$ E. $2\sqrt{3}-\pi$

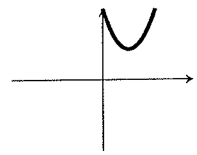


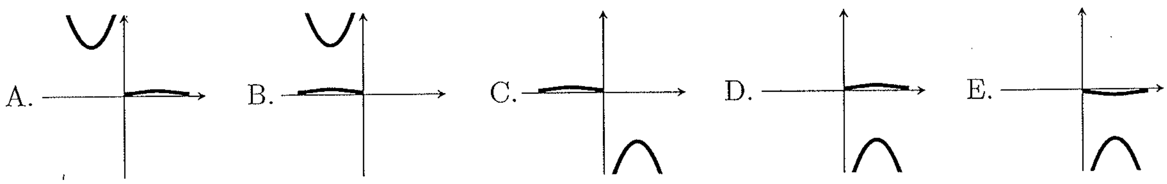
13. If $f(e^x) = \sqrt[3]{x}$ for $x > 1$, what is $f^{-1}(x)$?
    A. $(\ln x)^3$ B. $3\ln x$ C. e^{x^3} D. e^{3x} E. $\sqrt[3]{e^x}$

14. Determine the sum $1+\dfrac{1}{1+\sqrt[3]{x}}+\dfrac{1}{(1+\sqrt[3]{x})^2}+\dfrac{1}{(1+\sqrt[3]{x})^3}+\cdots$ where $x<-8$.

A. $\dfrac{-1}{\sqrt[3]{x}}$ B. $\dfrac{1}{\sqrt[3]{1+x}}$ C. $\dfrac{1}{1+\sqrt[3]{x}}$ D. $\dfrac{-1}{\sqrt[3]{x}-1}$ E. $1+\dfrac{1}{\sqrt[3]{x}}$

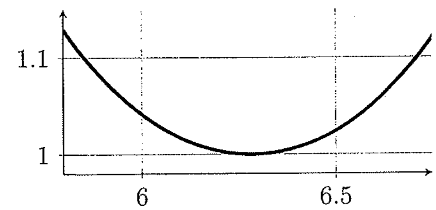
15. Consider the graph of the function f shown to the right. Which of the following images shows the graphs of $y=-f(x)$ and $y=\frac{1}{f(x)}$?





16. If we measure x in radians, which trigonometric function is graphed on the right?

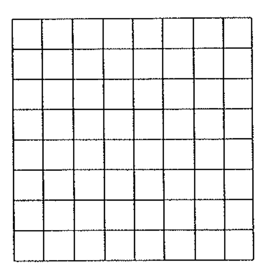
A. $\cos x$ B. $\cot x$ C. $\csc x$ D. $\sec x$ E. $\tan x$



17. If $x=\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{\cdots}}}}}}$, then x is

A. -3 B. 12 C. 4 D. $4\sqrt{3}$ E. ∞

18. Jack and Jill decided to paint the grid shown to the right. To paint it, they took turns starting with Jack. On his turn, Jack painted half of the unpainted squares blue. On her turn, Jill painted half of the unpainted squares green. They repeated this until they were down to just two squares left; at which time they each painted one of the squares. What is the ratio of the number of blue squares to the number of green squares?



A. 3 to 1 B. 1 to 1 C. 43 to 21 D. 21 to 11 E. 17 to 15

19. Define a function on the the integers by $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{otherwise.} \end{cases}$ For example, $f(5) = 16$, so $f(f(5)) = 8$, $f(f(f(5))) = 4$, $f(f(f(f(5)))) = 2$ and $f(f(f(f(f(5))))) = 1$. After applying f five times to 5, we got 1. If we start with 9, how many times must we apply f to get 1?

A. 5 B. 7 C. 9 D. 13 E. 19

20. How many non-congruent right triangles are there with integer length legs a, b and hypotenuse $b+1$ where none of these lengths is greater than 50?

A. 4 or fewer B. 5 C. 6 D. 7 E. 8 or more

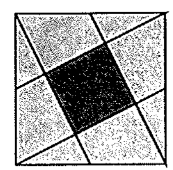
21. Lois has all of the answers to this test and wants to sell these to some of her 800 friends for x cents each. The number of friends that will buy them at x cents each is the minimum of 800 and $100 + 60x - x^2$. What price should Lois charge in order to make the most money?

A. 16 ¢ B. 30 ¢ C. 41 ¢ D. 42 ¢ E. 44 ¢

22. Victor ran 20 meters northwest. Then Victor turned and ran six meters southwest. Finally he ran eight meters east. How far is Victor from where he started? Approximate your answer to the nearest hundredth.

A. 9.90 m B. 10.38 m C. 10.44 m D. 14.35 m E. 18.76 m

23. Bisect the sides of a square, and draw "diagonals" as indicated on the right. The area of the small dark square is what proportion of the area of the original square?



A. 0.20 B. 0.24 C. 0.25 D. 0.28 E. 0.30

24. A polynomial p is given by $p(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. When $p(x)$ is divided by $x+1$, the quotient is $q(x) = gx^4 + hx^3 + jx^2 + kx + \ell$ and the remainder is 7. What is $a - b + c - d + e - f$?

A. 7 B. ℓ C. $-\ell$ D. -7 E. 0

25. How many real numbers have the property that the sum of the number and its multiplicative inverse equals the product of the number and its additive inverse?

A. zero B. one C. two D. three E. more than three

26. The voltage in an AC circuit can be modeled with sinusoids of the form $V(t) = Asin(Bt+C)+D$. Which of the following voltage functions represents a circuit whose voltage peaks at 5 volts and has an average voltage of 2 volts?

A. $V(t) = 3\sin(t) + 2$

B. $V(t) = 5\sin(t) + 2$

C. $V(t) = 5\sin(2t)$

D. $V(t) = 5\sin(t + 2)$

E. $V(t) = 2\sin(5t)$

27. Which of the following is a characteristic of the function $f(x) = 10^{200}x^2 - 2 \cdot 10^{100}x + 3$?

A. $f(x)$ always increases as x increases.

B. The equation $f(x) = 0$ has no real solution.

C. $f(x)$ has a maximum value when x is negative.

D. $f(x)$ has a minimum value of -1

E. The graph of $y = f(x)$ is an ellipse.

28. When evaluated, how many digits are in 7^{5^4}?

A. 263 B. 264 C. 528 D. 529 E. 625

29. If $x+2y+3z = 12$, $3x+2y+z = 8$ and $4x+5y+6z = 27$, what is $x+y+z$?

A. 47 B. 23 C. 18 D. 5 E. 8/27

30. Determine the equation of the parabola with focus $(2, 4)$ and directrix $y = 6$.

A. $y + 5 = -4(x - 2)^2$

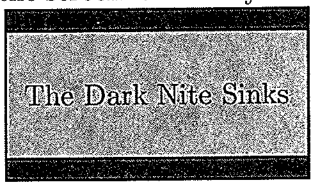
B. $y = -4x^2 + 16x - 11$

C. $y = \frac{1}{4}x^2 - x + 6$

D. $y = \frac{-1}{4}x^2 + x + 4$

E. $y = \frac{-1}{4}x^2 - x + 4$

31. High-definition television screens have an aspect ratio (ratio of width to height) of 16:9. Wide-screen films (aspect ratio 2.39:1) are shown on such television sets using the full width of the screen, leaving black bars on the top and bottom. If it can be determined, what percentage of the screen is actually used to show the movie? Round your answer to the nearest whole percent.



A. 74%

B. 27%

C. 95%

D. 85%

E. Cannot be determined without knowing the screen size of the television.

32. How many solutions does $\sin\left(\frac{1}{x}\right) = 0$ have in the interval $(0.01, 0.1)$?

A. at most ten

B. between eleven and twenty

C. between twenty-one and forty

D. between forty-one and eighty

E. more than eighty

33. The coordinates of the points A, B and C are respectively (5,5), (2,1) and $(0, y)$. The value of y which makes the length of the path $\overline{AC} + \overline{CB}$ as small as possible is

A. 3 B. $4\frac{1}{2}$ C. $3\frac{6}{7}$ D. $1\frac{2}{5}$ E. $2\frac{1}{7}$

34. Define $x = 200^{300^{400}}$, $y = 400^{300^{200}}$ and $z = 300^{400^{200}}$. Which of the following is true?

A. $y < x < z$ B. $y < z < x$ C. $z < x < y$ D. $z < y < x$ E. $x < z < y$

35. Suppose a function $f(x)$ satisfies $f(nm) = nf(m) + mf(n)$ for all integers n and m. What is $f(-1)$?

A. 0 B. -1 C. 1 D. m E. Not enough information to determine

36. The function $E(x)$ is an even function. Which of the following is an odd function?

A. $E(x) - E(-x)$ B. $E(x)^3$ C. $\sin(E(x))$ D. $\dfrac{3E(x)}{E(x)^3+1}$ E. $2015E(x)+1$

37. If $\left(x + \dfrac{1}{x}\right)^2 = 3$, then $x^3 + \dfrac{1}{x^3}$ equals

A. 0 B. $3 + \dfrac{1}{3}$ C. $3\sqrt{3}$ D. $-3\sqrt{3}$ E. $6\sqrt{3}$

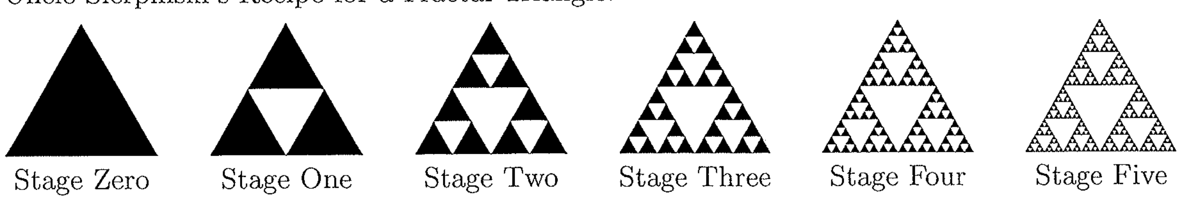
38. Hextpah has an open vat which contained 1000 kilograms of diet cola. She was dismayed to find out diet cola was 99.74% water. She left the vat uncovered until enough water evaporated for the cola to be only 89.62% water (which is the same as regular cola). About how many kilograms does it weigh now? Round your answer to the nearest tenth.

A. 896.2 B. 893.9 C. 293.9 D. 90.1 E. 25.0

39. The binary operation $\bowtie$ is defined by $x \bowtie y = e^{x+y}(\cos e^y + i \sin e^y)$ where $i^2 = -1$ and the absolute value of the complex number $a + bi$ is defined by $\sqrt{a^2+b^2}$. Which of the following has the largest absolute value?

A. $5 \bowtie 500$ B. $140 \bowtie 350$ C. $-(300 \bowtie 220)$ D. $(-200) \bowtie 640$ E. $(-100) \bowtie 600$

40. Uncle Sierpiński's Recipe for a Fractal Triangle:



Start at "stage zero" with a solid (filled) equilateral triangle with area one square unit. To find each subsequent stage, subdivide each solid triangle into four smaller congruent equilateral triangles and remove the central one. Continue this process forever. What is the total area, in square units, of all of the solid triangles from all of the stages (Zero, One, Two, . . .)?

A. 2 B. 4 C. $\dfrac{9}{4}$ D. $\dfrac{15}{2}$ E. The total area will be infinite.