TENNESSEE MATHEMATICS TEACHER' ASSOCIATION

SIXTY-FOURTH ANNUAL MATHEMATICS CONTEST

2022

Algebra II/Integrated III

Prepared by:

Joseph Fox
Aquinas College
Nashville, TN

Reviewed by:

Mathematics Faculty
Austin Peay State University
Clarksville, TN
University of TN at Martin
Martin, TN

Scoring formula: 4 x (Number Right) – (Number Wrong) + 40

Directions:

Do not open this booklet until you are told to do so

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

2022 TMTA Algebra 2/Integrated III

1. Suppose $f(x) = \sqrt{x-3}$ and $g(x) = \frac{1}{1-x^2}$. What is the domain of $g \circ f$?

   (a) $[3, \infty)$

   (b) $(-\infty, 4) \cup (4, \infty)$

   (c) $(-\infty, \infty)$

   (d) $(-\infty, -1) \cup (-1,1) \cup (1, \infty)$

   (e) $[3,4) \cup (4, \infty)$

2. The number of bacteria on a wet sponge is given by $B = 2500e^{0.11t}$, where t is the number of hours that have passed since you finished doing the dishes at 4:30pm. To the nearest minute, when will the population size have doubled?

   (a) 10:00pm (b) 10:48pm (c) 11:47pm (d) 1:13am

   (e) 3:45am

3. What is the equation of the circle which is tangent to the y-axis, the x-axis, and the circle $(x-6)^2 + (y-2)^2 = 4$?

   (a) $(x-2)^2 + (y-2)^2 = 4$

   (b) $(x-6)^2 + (y-2)^2 = 9$

   (c) $(x-6)^2 + (y-6)^2 = 4$

   (d) $x^2 + (y-2)^2 = 4$

   (e) $(x+2)^2 + (y-2)^2 = 36$

4. Which of the following x-values has the property that $x^2 = i$?

   (a) $x = -1$ (b) $x = 1 + i$ (c) $x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

   (d) $x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (e) $x = -i$

5. Suppose a is a positive number. In terms of a, what are the x-intercepts of the graph of $y = -2|x - 3| + a$?

(a) $\left(\frac{a}{2} + 3, 0\right)$ and $\left(-\frac{a}{2} + 3, 0\right)$

(b) $(0, 0)$ and $(a + 6, 0)$

(c) $(a - 3, 0)$ and $(a + 3, 0)$

(d) $\left(\frac{a}{2} + 3, 0\right)$ and $\left(\frac{a}{2} - 3, 0\right)$

(e) $(a + 3, 0)$ and $(-a + 3, 0)$

6. What is the solution set of the inequality below?

$$\frac{\frac{1}{x} + 1}{x + 1} < x$$

(a) $(-\infty, 1)$ (b) $(-1, 0) \cup (1, \infty)$ (c) $(0,1)$

(d) $(1, \infty)$ (e) $(-\infty, 0)$

7. Suppose $(-2.5, -2, -1.5, -1, \ldots)$ is the beginning of an arithmetic sequence. What is the 501^{st} term of the sequence?

(a) 247 (b) 247.5 (c) 248 (d) 248.5 (e) 249

8. Suppose b is a real number and let $f(x) = \log_3(x - b)$. If the graph of f passes through the point $(10,2)$, what is the domain of f ?

(a) $(0, \infty)$ (b) $(1, \infty)$ (c) $(2, \infty)$ (d) $(-\infty, 0)$ (e) $(-\infty, 10)$

9. To which of the following expressions does $(81x^8y^4)^{-3/4}\left(\frac{2}{3}x^{-1/2}y^3\right)^{-2}$ simplify?

(a) $\frac{9y^6}{\sqrt{x}}$ (b) $\frac{12}{x^2y^9}$ (c) $\frac{9\sqrt{x}}{4y^6}$

(d) $\frac{18}{\sqrt[3]{x}y^3}$ (e) $\frac{1}{12x^5y^9}$

10. Suppose A is a 3×5 matrix, B is a 3×3 matrix, and C is a 5×5 matrix. Which of the following matrix computations can be performed?

(I) $A \cdot B$ (II) $B \cdot A + A \cdot C$ (III) $A^T \cdot B$ (IV) $(B \cdot A)^T \cdot C$

(a) (I) and (III) (b) (II) and (III) (c) (II) only

(d) (II), (III), and (IV) (e) (I), (II), (III), and (IV)

11. Which of the following is an asymptote of the function $f(x) = \frac{x^2-x}{x-1} + 2$?

(a) $x = 1$ (b) $y = 0$ (c) $y = x$ (d) $y = x + 2$ (e) $y = 2$

12. A certain triangle has a side of length 10 and another of length 7. The angle between these two sides is $36°$. What is the triangle's area? (Choose the answer which is closest to the true value.)

(a) 20.57 (b) 28.32 (c) 35 (d) 41.14 (e) 70

13. Suppose someone records a funny video of their cat and an hour later sends it to a friend. An hour later, that friend sends it to two new friends. After another hour, those two friends each send it to three new friends. After another hour, those six friends each send it to four new friends. This pattern continues throughout the day. Twelve hours after the original video was first sent, how many people in total have received the funny cat video (not counting the person who originally recorded it)?

(a) 2^{12}

(b) $(1!) \times (2!) \times (3!) \times \cdots \times (12!)$

(c) $1! + 2! + 3! + 4! + \cdots + 13!$

(d) $12!$

(e) $1 + 2 + 3 + \cdots + 12$

14. What is the inverse of the function $f(x) = 2\ln(x^3 + 1) - 4$?

(a) $f^{-1}(x) = \frac{1}{2\ln(x^3+1)-4}$

(b) $f^{-1}(x) = \frac{1}{2}\ln\left(\sqrt[3]{x} - 1\right) + 4$

(c) $f^{-1}(x) = \sqrt[3]{e^{(x+4)/2} - 1}$

(d) $f^{-1}(x) = e^{\sqrt[3]{(x+4)/2-1}}$

(e) $f^{-1}(x) = \sqrt[3]{\frac{x+4}{2} - 1}$

15. After a certain type of medication enters your body, it reacts in such a way that during each hour that passes, 7% of the medication that was present at the beginning of the hour is no longer present. If 40 mL of the medication is initially ingested, how much will remain after 48 hours?

(a) 0.80 mL (b) 1.23 mL (c) 5.61 mL (d) 7.89 mL

(e) 14.48 mL

16. Let a be a real number. Which of the following functions definitely does *not* have a vertical asymptote at $x = a$?

(a) $f(x) = \frac{x+2}{x-a}$ (b) $f(x) = \frac{x^4-a^4}{x-a}$ (c) $f(x) = \frac{x+a}{x-a}$

(a) $f(x) = \frac{x^3+a^3}{x-a}$ (e) $f(x) = \frac{x^2-a}{x-a}$

17. In a certain group of elk, 5% of the animals are sickly. If an elk is sickly, then the possibility of being eaten by wolves is 33%. The probability of a healthy elk being eaten by wolves is 0.66%. What is the probability that an elk chosen at random from the heard will be eaten by wolves? (Choose the answer which is the closest to the true value.)

(a) 11% (b) 1% (c) 38.7% (d) 2.3% (e) 87%

18. What is the value of i^{2019} ?

(a) 1 (b) -1 (c) i (d) $-i$ (e) 0

19. What is the value of $18 - 16 \div 8^{-2/3}$?

(a) -46 (b) 8 (c) $18 - \sqrt[3]{4}$ (d) 14 (e) $\frac{1}{2}$

20. Which of the following is a term in the expansion of $\left(\frac{1}{x} - x\right)^5$?

(a) x^5 (b) $\frac{1}{x}$ (c) x^3 (d) $\frac{10}{x}$ (e) $-5x^3$

21. For a sewer line to comply with the building code, it has to achieve a "fall" of at least ¼-in-1. This means that for each foot of horizontal distance covered by the line, the vertical distance of the line below the horizontal must increase by ¼ of an inch. Suppose a sewer line is set at an angle of α degrees below the horizontal. Which of the following is the smallest value of α which would comply with the building code?

(a) $\alpha = 0.5$ (b) $\alpha = 1.1$ (c) $\alpha = 0.021$

(d) $\alpha = 1.2$ (e) $\alpha = 2.3$

22. Suppose $\cos(\theta) = 0.25$. Which of the following *must* be true?

(a) $\sin(\theta) = \frac{\sqrt{15}}{4}$

(b) $\tan(\theta) = \sqrt{15}$

(c) $\cos\left(\theta - \frac{\pi}{2}\right) = -\frac{\sqrt{15}}{4}$

(d) $\cos(-\theta) = 0.25$

(e) $\sin(-\theta) = -\frac{\sqrt{15}}{4}$

23. A boat traveled 60 miles down a river in the same direction as the current in 3 hours. It then turned around and traveled against the current back to its original starting point. The return trip took 5 hours. What is the speed of the river's current?

(a) 2 mph (b) 4 mph (c) 5 mph (d) 12 mph (e) 20 mph

24. Suppose a and b are real numbers. What is the y-intercept of the line that passes through the points (a, b) and (a^2, b^2) ?

(a) $\frac{b(b-1)}{a(a-1)}$ (b) $\frac{b^2}{a^2}$ (c) b (d) $\frac{b(a-b)}{a-1}$ (e) $\frac{a(a-1)}{b+1}$

25. A train leaves Boston at 9am and travels at a constant rate of 35 mph until it reaches its first stop in Hartford, CT, at noon. The train leaves Hartford at 12:30pm and travels 115 miles, reaching its final destination of New York City at 3pm. What was the average speed of the train during the entire trip? (Choose the answer closest to the true value.)

(a) 27 mph (b) 35 mph (c) 36.67 mph

(d) 40.5 mph (e) 46 mph

26. Suppose the letters a, b, c, d, e and f represent real numbers. What is the maximum number of zeros of the function $g(x) = ax^3(b - cx^4 + dx^5) + ex + f$?

(a) 1 (b) 3 (c) 12 (d) 13 (e) 15

27. When you roll a 6-sided die 3 times, what is the probability that you roll a 2 at least once?

(a) $\frac{91}{216}$ (b) $\frac{125}{216}$ (c) $\frac{1}{216}$ (d) $\frac{1}{6}$ (e) $\frac{3}{6}$

28. A parabola passes through the points $(0,2)$, $(4,5)$, and $(8,-2)$. What is the maximum y-value among all the points on this parabola?

(a) $\frac{16}{5}$ (b) $\frac{26}{5}$ (c) 5 (d) 8 (e) $\frac{32}{5}$

29. For what values of a and b does the graph of the function $f(x) = a\log_3(bx + 1)$ have a vertical asymptote at $x = 2$ and pass through the point $(-4,1)$?

(a) $a = 2, b = 2$ (b) $a = \frac{1}{2}, b = \frac{1}{2}$ (c) $a = -2, b = \frac{1}{2}$

(d) $a = 1, b = -\frac{1}{2}$ (e) $a = 2, b = -2$

30. For the following system of equations, for what value of $b - c$ is it possible that there are infinitely many solutions for x, y, and z?

$$\begin{aligned} 2ax + by &= 4 \\ y + 3z &= 1 \\ ax + by + cz &= 0 \end{aligned}$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

31. A hanging sign is supported by three wires attached to a wall. During the course of a year, each wire has a 10% chance of breaking. If any two of the wires breaks, the sign will fall. What is the probability that the sign will make it through the year without falling?

(a) 29.8% (b) 70.2% (c) 2.8% (d) 97.2% (e) 98.4%

32. A circle C has radius 4 and is centered at the point $(2,4)$. Suppose C is transformed as follows: While the center is held fixed, the area of C is doubled. Then the enlarged version of C is reflected across the y-axis. What is the equation of the circle that results from these transformations?

(a) $(x+2)^2 + (y+4)^2 = 8$

(b) $(x+2)^2 + (y-4)^2 = 32$

(c) $(x+2)^2 + (y-4)^2 = 64$

(d) $(x-2)^2 + (y+4)^2 = 64$

(e) $(x-2)^2 + (y-4)^2 = 16$

33. Suppose l_1 and l_2 are parallel lines. The equation of l_1 is $x - 2y = -3$ and the y-intercept of l_2 is 1. What is the perpendicular distance between l_1 and l_2?

(a) $\frac{\sqrt{5}}{5}$ (b) $\frac{\sqrt{5}}{2}$ (c) 2 (d) 0.67 (e) $\frac{\sqrt{3}}{2}$

34. The great Ancient Greek mathematician Diophantus lived a long life. His youth lasted the first 1/6 of his life. The next 1/12 of his life was spent growing a beard. After another 1/7 of his life, he married. Five years later, his son was born. Sadly, his son died at an age that was exactly half of Diophantus' final age. Four years after his son died, Diophantus' life came to an end as well. If D represents Diophantus' age when he died and S represents his son's, which of the following systems of equations correctly models the statement above?

(a) $\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{7}\right)D+\frac{1}{2}S=D,\ S=\frac{1}{2}D$

(b) $\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{7}\right)D+S+9=D,\ S=\frac{1}{2}D$

(c) $S+4=D,\ S=\frac{1}{2}D$

(d) $\frac{1}{6}D+\frac{1}{12}D+2S+5=S+4,\ S=2D$

(e) $\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{7}+\frac{1}{5}+\frac{1}{4}\right)D=D,\ S=\frac{1}{2}D$

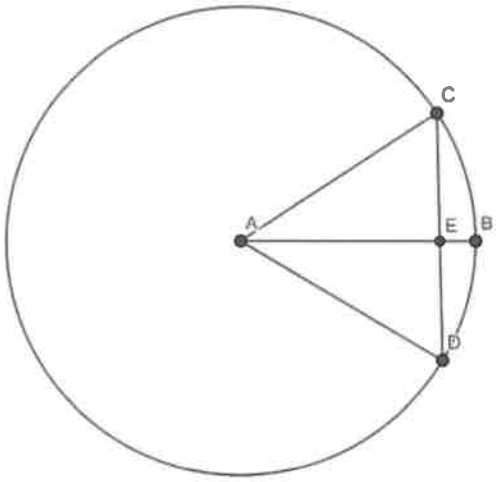
35. What is the sum of the solutions of the equation $\sqrt{x}=\sqrt[3]{x^2-2x}$?

(a) 0 (b) 1 (c) 2 (d) 4 (e) 5

36. Find the distance from the origin to the point on the line $x+2y=8$ which is closest to the origin. (Choose the answer which is closest to the true value.)

(a) 2.13 (b) 3.58 (c) 3.87 (d) 4.04 (e) 4.29

37. In the diagram below, point A is the center of the circle, $\overline{AB}$ is perpendicular to $\overline{CD}$, $\angle CAD = 50°$, and the radius of the circle is 12. How long is $\overline{EB}$? (Choose the answer which is closest to the true value.)



(a) 1.12 (b) 1.88 (c) 2.37 (d) 2.61 (e) 2.75

38. Suppose b is a real number. Which of the following equations could *possibly* have complex solutions?

(a) $x^2 + bx - b^2 = 0$

(b) $x^2 + (b+1)\mathrm{x} - 1 = 0$

(c) $x^2 + bx - b = 0$

(d) $b^2x + bx - 1 = 0$

(e) $(b-1)x^2 + bx + 1 = 0$

39. What is the product of the solutions of the equation $(\log_2 x)^2 + \log_2(x^{-1}) = 6$?

(a) $\frac{1}{4}$ (b) 2 (c) 4 (d) 8 (e) 32

40. Suppose a 40-question math test has a multiple choice format with 5 choices per question. (Maybe you've taken such a test before.) Questions 1-20 are relatively easy – you can eliminate 3 of the choices for each question, making it a toss-up between the other 2. For questions 21-30, you can eliminate 2 choices, but the other 3 seem equally likely to you of being correct. For questions 31-40, you can only eliminate 1 choice, leaving the other 4 as potentially equally likely. What is the probability that, if you make random guesses among the choices you did not eliminate, you'll get a perfect score?

(a) $\left(\frac{1}{5}\right)^{40}$

(b) $\left(\frac{1}{2}\right)^{20} + \left(\frac{1}{3}\right)^{10} + \left(\frac{1}{4}\right)^{10}$

(c) $\left(\frac{1}{2}\right)^{20} \times \left(\frac{1}{3}\right)^{10} \times \left(\frac{1}{4}\right)^{10}$

(d) 50%

(e) $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$